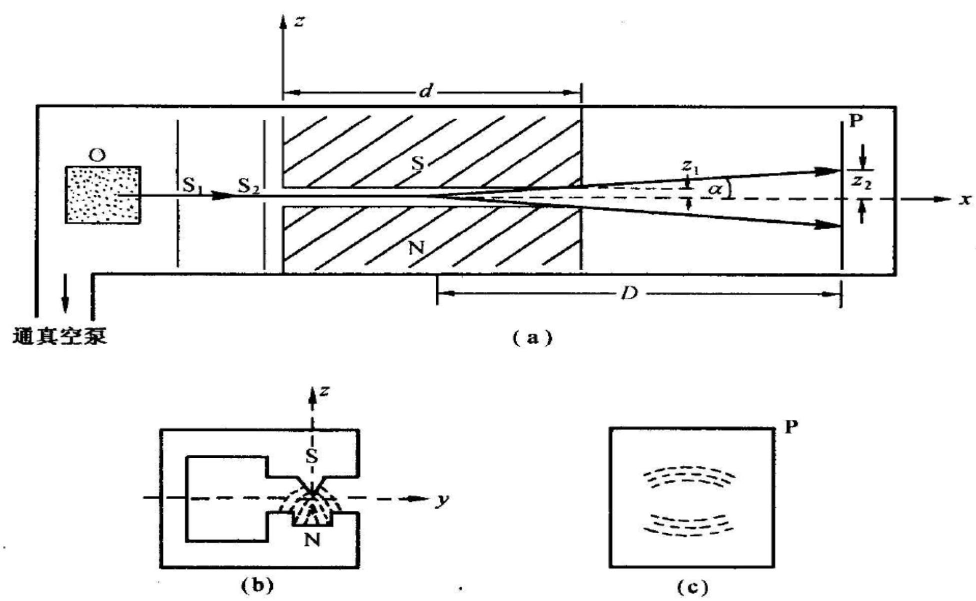
Fundamentals of Spectroscopy for Optical Remote Sensing
Homework #5 (Atomic Structure)

1. Assume an atom has the total spin angular momentum number S=1 and the total orbital angular momentum number L = 3 of electrons. If there exists L-S coupling in this atom, please write down (1) all possible total angular momentum number J for the electrons, (2) the atomic energy state terms, and (3) the possible magnetic quantum number m_J for each J.
2. For atomic states $^2S_{1/2}$, $^2P_{1/2}$, $^2P_{3/2}$, $^2D_{3/2}$, and $^2D_{5/2}$, please (1) calculate the Lande g-factor g_J and then (2) derive all possible z-components $m_J g_J$ for each state. You may list them in a table.
3. For an atom in the energy state $^2D_{3/2}$, please drive the atomic magnetic moment $\vec{\mu}$ and the possible values of its projection in z direction, i.e., μ_z. You may express the results in the unit of Bohr magneton μ_B.
4. In the Stern-Gerlach experiment (see figure below), a narrow beam of silver atoms in the ground state ($^2S_{1/2}$) pass through an inhomogeneous magnetic field (perpendicular to the atomic beam x-direction) and strike on a screen. The magneton longitudinal range is d = 10 cm, the range from the magneton's center to the screen is D = 25 cm. The silver atoms have speed of 400 m/s, and the pattern splitting on the screen is 2.0 mm. How much is the gradient of the magnetic field along z-direction (i.e., $\frac{\partial B_z}{\partial z}$)?

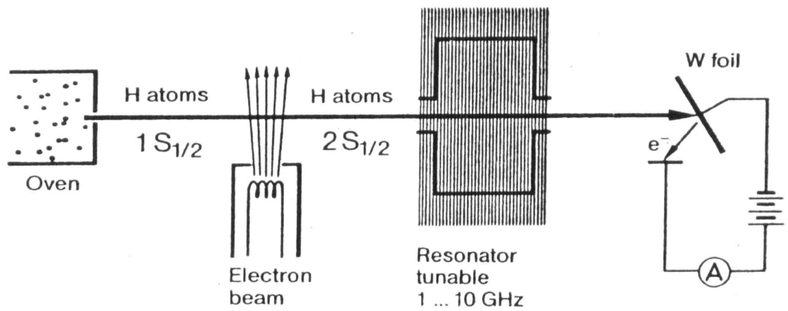


5. Fine structure and hyperfine structure of hydrogen atoms (given by Dirac equation − relativity quantum mechanics)
   (1) Calculate the energy levels of n = 1, 2, and 3 for hydrogen atom when only considering the electrostatic interaction (Coulomb force) between the nucleus and the electron. The potential energy at infinite is set to zero.
   (2) Calculate the fine structure for each energy level of n = 1, 2, and 3. The relativity mass correction (ΔE_m), the Darwin term (ΔE_d), and the spin-orbit coupling (ΔE_{ls}) should be calculated, and then the sum ΔE of these three terms should be derived. Please show the procedures how you do the calculation, and then put the results into a table.

(3) Draw a diagram to show the coarse and fine structures of the energy levels (only the final results for the fine structure). Please mark in the energy level diagram the state symbols and the shift relative to the coarse energy levels (n = 1, 2, 3). Use Joule and MHz as the energy unit.

(4) Derive the hyperfine structure for the hydrogen ground state, and calculate the hyperfine splitting between the hyperfine states. Convert the energy split to frequency and wavelength.

6. Lamb shift refers to the energy shift that causes the separation of $2^2P_{1/2}$ and $2^2S_{1/2}$ for n = 2 energy levels of hydrogen atom. It also causes the separation of $3^2P_{1/2}$ and $3^2S_{1/2}$, and of $3^2D_{3/2}$ and $3^2P_{3/2}$. Lamb shift cannot be explained by relativity quantum mechanics but by quantum electrodynamics (QED). Lamb shift was discovered by Lamb and Retherford in 1947 with their radiofrequency spectroscopy experiment: the atomic-beam magnetic-resonance method (see below).



An oven of H_2 gas is heated to 2500 K so H_2 molecules are resolved into hydrogen atoms. These atoms escape the oven through a small hole and form a H atomic beam. Entering the vacuum area, the H atomic beam is bombarded by accelerated electrons with energy larger than 10.2 eV, which excite H atoms from its ground state 1 $^2S_{1/2}$ to n = 2 energy levels ($2^2P_{3/2}$, $2^2P_{1/2}$ or $2^2S_{1/2}$). Because the transition from $2^2S_{1/2} \rightarrow 1^2S_{1/2}$ is forbidden, $2^2S_{1/2}$ is a metastable state with a very long lifetime (0.14 s). But $2^2P_{1/2}$ has very short lifetime (1.595 ns) due to the transition from $2^2P_{1/2} \rightarrow 1^2S_{1/2}$. Consequently, after certain time of flight, all $2^2P_{3/2}$ amd $2^2P_{1/2}$ atoms disappear and only H atoms in $2^2S_{1/2}$ and $1^2S_{1/2}$ states are left in the atomic beam. When H atoms strike on a detector made of tungsten (W) at the end of the beam, because the work function of W is less than 10.2 eV, the $2^2S_{1/2}$ H atoms can kick out electrons from the detector to generate a current but $1^2S_{1/2}$ H atoms cannot. Therefore, such a detector is only to detect the H atoms in the metastable state $2^2S_{1/2}$. Before the atomic beam arrives at the detector, the atomic beam will go through a microwave radiation under a static magnetic field. If the microwave is tuned to excite the $2^2S_{1/2}$ atoms to $2^2P_{3/2}$, then the number of $2^2S_{1/2}$ atoms reaching the detector will be reduced thus a decrease in the generated current. By this method, the energy separation between $2^2P_{3/2}$ and $2^2S_{1/2}$ can be measured.

(1) Dirac's theory predicts that $2^2P_{1/2}$ and $2^2S_{1/2}$ are degenerated and the separation between $2^2P_{3/2}$ and $2^2S_{1/2}$ should be ~10,970 MHz. However, the measured resonance frequency was ~1058 MHz lower than expectation. Which energy level is higher between $2^2P_{1/2}$ and $2^2S_{1/2}$?

(2) H atomic H_α line is the transition from n = 3 to n = 2 energy levels. Please use the fine structure you derived from Problem 5, and apply the following selection rules $\Delta L = \pm 1$ and $\Delta J = 0, \pm 1$ to figure out : How many fine lines does the H_α line have if Lamb shift is ignored? How many fine lines does the H_α line actually consist of if Lamb shift is considered? Please list all of the fine transition lines of H_α line.

HW #5 is due on Tuesday, October 8th, 2013 in class.

Extra exercise problem --

7. Consider a system of angular momentum J. We confine ourselves in this exercise to a three-dimensional subspace, spanned by the three kets |+1>, |0>, |-1>, common eigenstates of J^2 (eigenvalue $2\hbar^2$) and J_z (eigenvalues $+\hbar$, 0, $-\hbar$). The Hamiltonian H_0 of the system is

$$H_0 = aJ_z + \frac{b}{\hbar}J_z^2$$

where a and b are two positive constants, which have the dimensions of an angular frequency. What are the energy levels of the system? For what value of the ratio b/a is there degeneracy?