Fundamentals of Spectroscopy for Optical Remote Sensing

Homework #2 (Quantum Mechanics)

1. In lidar remote sensing application, sometimes our return photon signal is so weak that you can practically consider the photons coming back one by one. However, when a Fabry-Perot etalon is used in the lidar receiver, it actually works as a narrowband daytime filter to pass the wanted frequency while rejecting lots of unwanted frequencies. Could you explain this single-photon Fabry-Perot etalon phenomenon from Quantum Mechanics point of view? (You may check optics book or on the web about how a Fabry-Perot etalon works.)

2. Please summarize the postulates and principles of Quantum Mechanics in your own words, including QM state, operator, measurement, eigenvalue equation, principle of superposition, principle of motion, principle of uncertainty, mean value, representations, etc.

3. In a one-dimensional problem, consider a particle whose wave function is

$$\psi(x) = N\frac{e^{ip_0 x/\hbar}}{\sqrt{x^2 + a^2}}$$

where a and p_0 are real constants and N is a normalization coefficient.
(1) Determine N so that $\psi(x)$ is normalized.
(2) The position of the particle is measured. What is the probability of finding a result between $-\frac{a}{\sqrt{3}}$ and $+\frac{a}{\sqrt{3}}$?
(3) Calculate the mean value of the momentum of a particle that has $\psi(x)$ for its wave function.

4. The energy operator for harmonic oscillator is

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}\kappa\hat{x}^2 = \frac{1}{2m}\left(\hat{p}^2 + m^2\omega_0^2\hat{x}^2\right),$$

where $\omega_0 = \sqrt{\kappa/m}$ is the intrinsic angular frequency of the harmonic oscillator. The normalized wave function for the ground state of the oscillator is given by

$$\psi_0(\xi) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4} e^{-\xi^2/2},$$

where $\xi = \left(\sqrt{m\omega_0/\hbar}\right)x$. (1) Please compute the mean values of the oscillator's momentum and energy. (2) Is this wave function an eigen function of the energy operator $\hat{H}$ (also called Hamilton operator)? If yes, how much is the eigen value?

5. For a particle moving within a range of x =0 to x = a, its normalized wave function is given by

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right), & 0 < x < a \\ 0, & elsewhere \end{cases}$$

Is this wave function an eigen function of momentum? Is it an eigen function of kinetic energy? If so, how much is the corresponding eigen value?

6. A particle's momentum and the position of itself have the commutation relation $[\hat{x},\hat{p}_x]=i\hbar$. However, the variables between different particles are commuted. In other words, the following relationship exists for particles 1 and 2:

$$[\hat{x}_i,\hat{p}_j]=i\hbar\delta_{ij}, \qquad (i,j=1,2).$$

Try to prove that the operators $\hat{x}_1-\hat{x}_2$ and $\hat{p}_1+\hat{p}_2$ are commuted with each other.

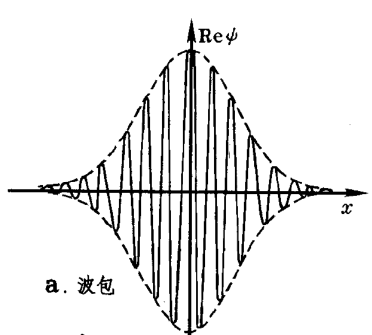
7. Please use the following conditions and steps to derive Heisenberg uncertainty relation.

(1) Assume a particle described by a wave packet (see Figues) has a Gaussian wave function $\psi(x)=Ae^{-ax^2/2}e^{+ip_0x/\hbar}$, where A is the normalization factor $A=(\pi/a)^{-1/4}$. The uncertainty of the coordinate x should be given by $\Delta x=\sqrt{\overline{x^2}}$. Try to derive its value from the formula $\overline{x^2}=\int_{-\infty}^{+\infty}\Psi^*(x)x^2\Psi(x)dx$.

(2) When expressed in the momentum representation, the wave function is given by $C(p)=\int_{-\infty}^{+\infty}\Psi(x)e^{-ipx/\hbar}\dfrac{dx}{\sqrt{2\pi\hbar}}=\sqrt{\dfrac{1}{a\hbar}}Ae^{-(p-p_0)^2/2a\hbar^2}$. The uncertainty of the momentum p should be given by $\Delta p=\sqrt{\overline{(p-p_0)^2}}$. Try to derive its value using the formula $\overline{(p-p_0)^2}=\int_{-\infty}^{+\infty}C^*(p)(p-p_0)^2C(p)dp$.

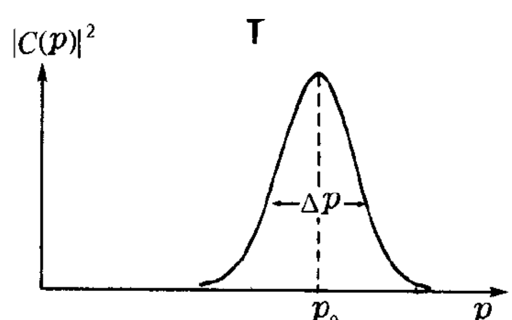
(3) After obtaining Δx and Δp, take the product of Δx and Δp to derive the Heisenberg uncertainty relation.

(4) Try to derive $\Delta p=\sqrt{\overline{(p-p_0)^2}}$ in the x representation using the formula $\overline{(p-p_0)^2}=\int_{-\infty}^{+\infty}\Psi^*(x)(\hat{p}-p_0)^2\Psi(x)dx$, where $\hat{p}=-i\hbar\dfrac{d}{dx}$. Do you get the same result as step (3)? If yes, why?



a. 波包

$|C(p)|^2$



b. 频谱