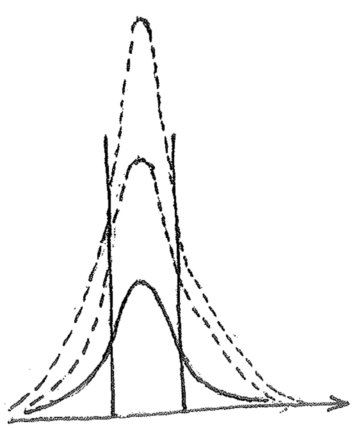
§7.7. Homogeneous and Inhomogeneous Line Broadening

* Homogeneous line broadening: three ways to describe it —

① If the probability $P_{ik}(\omega)$ of absorption or emission of radiation with frequency ω causing a transition $E_i \to E_k$ is equal for all the atoms or molecules of a sample that are in the same level E_i, we call the spectral line profile of this transition homogeneously broadened.

② Broadening processes have the same effect on all atoms in the sample, i.e., each atom has the same broadening.

③ Each atom contributes to the entire line profile spectrum, not just part of the spectrum.



Example: 3 atoms' contributions same $\delta\omega$, but 3 times of peak intensity.

Natural linewidth is a typical homogeneous broadening: in this case, the probability for emission of light with frequency ω on a transition $E_i \to E_k$

$$P_{ik}(\omega) = A_{ik}\, g_L(\omega)$$

is equal for all atoms in level E_i.
$g_L(\omega)$ is the normalized Lorentzian profile.

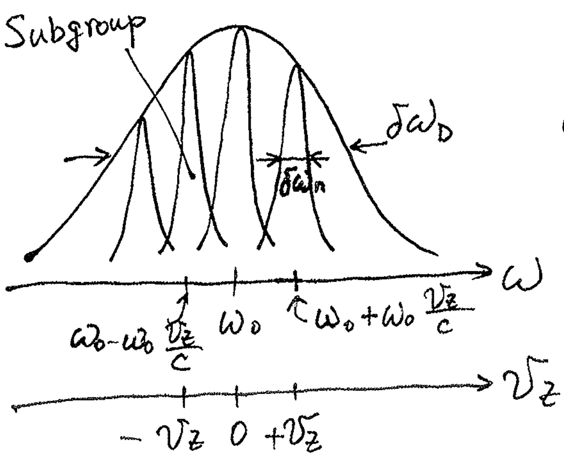
When we have N atoms in the sample, the overall line profile has peak intensity of N times single atom contribution, but the overall linewidth is equal to single atom linewidth γ_n.

* Inhomogeneous line broadening: nearly opposite to homogeneous broadening case:

① The probability of absorption or emission of radiation $E(\omega)$ is not equal for all atoms, but depends on certain conditions of the atoms.

② Each atom only contributes to part of the spectrum.

A typical example of inhomogeneous line broadening is the Doppler broadening. In this case, the probability of absorption or emission of radiation $E(\omega)$ is dependent on their velocity v.



We divide the atoms in level E_i into subgroups such that all atoms with a velocity component within $v_z \to v_z+\Delta v_z$ belong to one subgroup.

If we choose $\Delta v_z = \delta\omega_n/k$ where $\delta\omega_n$ is the natural linewidth, we may consider the frequency interval $\delta\omega_n$ to be homogeneous broadening of each subgroup inside the much larger inhomogeneous Doppler width. Thus, all atoms in the sub-group can absorb or emit radiation with frequency $\omega=\omega_0+v_z k$ because in the coordinate system of the moving atoms, this freq. is within the natural width $\delta\omega_n$ around ω_0.

Homogeneous broadening is usually Lorentzian line profile.
Inhomogeneous broadening like Doppler width is Gaussian shape.

Let us consider collision broadening:

① Inelastic collisions: result in a homogeneous Lorentzian line profile.

② Elastic Collisions: phase-perturbing collisions → homogeneous
velocity-changing collisions

$\begin{pmatrix} v_z \pm \Delta v_z \\ \text{Subgroup} \end{pmatrix} \longrightarrow \begin{pmatrix} v_z + u_z \pm \Delta v_z \\ \text{Subgroup} \end{pmatrix} \Rightarrow$ freq. $\omega \to \omega + k u_z$.

In Doppler-limited spectroscopy, the velocity-changing collisions do not affect the Doppler profile.

In Doppler-free spectroscopy,

if $T = \Lambda/\bar{v} > \tau_c$, population change caused by velocity-changing collisions is negligible.

if $T \ll \tau_c$, velocity-changing collisions cause homogeneous broadening of each subgroup, through shortening the effective interaction time of the atoms with the radiation field — move atoms out of resonance.

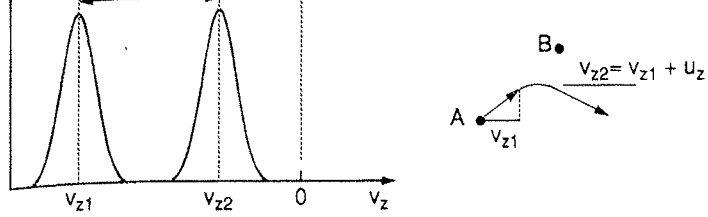
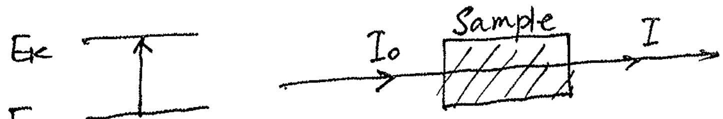


Fig. 3.21. Effect of velocity-changing collisions on the frequency shift of homogeneous subgroups within a Doppler-broadened line profile

§7.8. Linear and Nonlinear Absorption

From Eq.(369) and Eq.(384), we know that the intensity decrease dI of a wave with intensity I propagates along the z direction through an absorbing sample is

$$dI = -I\alpha\, dz = -I\,\sigma_{ik}\left[N_i - \frac{g_i}{g_k} N_k\right] dz \qquad (446)$$



where α is the absorption coefficient caused by transition $E_i \rightarrow E_k$

σ_{ik} is the absorption cross-section

N_i and N_k are the populations on E_i and E_k levels.

g_i and g_k are the degeneracy factors of E_i and E_k.

(1) Linear absorption:

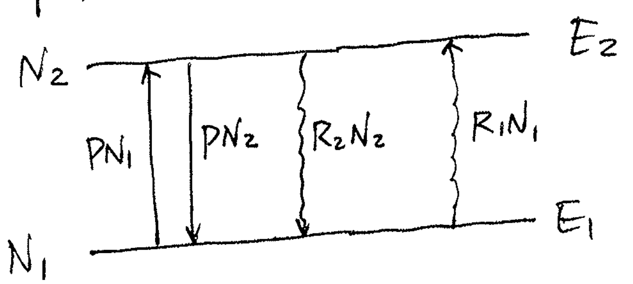
As long as the population densities N_i and N_k of the two levels E_i and E_k are not noticeably altered by the interaction with the radiation field (weak-signal), we can regard them as constant (independent of I). σ_{ik} is independent of I, thus, the absorption coefficient $\alpha = \sigma_{ik}\left[N_i - \frac{g_i}{g_k} N_k\right]$ is constant and independent of I. The absorbed intensity is then proportional to the incident intensity (linear absorption). Integration of Eq.(446) over the absorption path z gives Beer's law for linear absorption as given by Eq.(370):

$$I(z) = I_0\, e^{-\sigma_{ik}[N_i - (g_i/g_k)N_k]z} = I_0\, e^{-\alpha z} \qquad (447)$$

with $\alpha = \sigma_{ik}[N_i - (g_i/g_k)N_k]$. (448)

(2) Nonlinear absorption:

At larger intensity I, the density N_i of the lower state E_i can noticeably decrease while the upper state number density N_k increase (strong-field). This means that $N_i(I)$ and $N_k(I)$ are functions of I and therefore dI is no longer proportional to I (nonlinear absorption). In this case, while the absorption cross-section σ_{ik} is still independent of I, but the absorption coefficient α depends on I, caused by intensity-dependent populations on the lower and upper levels N_i and N_k.



Let us illustrate this nonlinear absorption by a simple example of a two-level system with population densities N_1 and N_2 and equal statistical weights $g_1 = g_2 = 1$. The total number density $N = N_1 + N_2$ of the two-level system is constant, if we exclude all decay channels to other levels than these two levels ($E_2 > E_1$).

The time derivations of N_1 and N_2 can be related to the Einstein coefficients $B_{21} = B_{12}$ for stimulated emission and absorption and A_{21} for spontaneous emission. If we allow additional collision induced transitions with probabilities C_{12} and C_{21} between the two levels, we obtain under stationary conditions $\left(\frac{dN_1}{dt} = -\frac{dN_2}{dt} = 0\right)$:

$$\frac{dN_1}{dt} = -\frac{dN_2}{dt} = -B_{12}\rho(\omega)N_1 + B_{21}\rho(\omega)N_2 + A_{21}N_2 + C_{21}N_2 - C_{12}N_1 = 0$$

$$\therefore \frac{dN_1}{dt} = -\frac{dN_2}{dt} = B_{12}\rho(\omega)(N_2 - N_1) + (A_{21} + C_{21})N_2 - C_{12}N_1 = 0 \quad (449)$$

Let $\Delta N = N_1 - N_2$; $N = N_1 + N_2 = \text{constant}$,

$R_2 = A_{21} + C_{21}$, $R_1 = C_{12}$, $P_1 = B_{12}\rho(\omega) = P_2 = B_{21}\rho(\omega) = P$ (450)

then the population difference ΔN can be written as

$$\Delta N = \frac{\Delta N_0}{1 + 2B_{12}\rho(\omega)/(R_1 + R_2)} = \frac{\Delta N_0}{1 + S} \quad (451)$$

where $\Delta N_0 = \frac{R_2 - R_1}{R_2 + R_1} N$ (452) is the population difference for $\rho(\omega) = 0$. ($\Delta N_0 = N_1^0 - N_2^0$)

Define the saturation parameter:

$$S = \frac{2B_{12}\rho(\omega)}{R_1 + R_2} = \frac{B_{12}\rho(\omega)}{(R_1 + R_2)/2} = \frac{B_{12}\rho(\omega)}{\bar{R}}, \quad (453)$$

where $\bar{R} = (R_1 + R_2)/2$. S represents the ratio of the induced transition probability $B_{12}\rho(\omega)$ to the mean relaxation probability $\bar{R}$. If the only relaxation process is spontaneous emission (i.e., $R_1 = 0$, $R_2 = A_{21}$), then the saturation parameter S yields the ratio of stimulated to spontaneous transition rates: $S = \frac{B_{12}\rho(\omega)}{A_{21}/2}$.

For $S = 1$, the population difference ΔN drops to one half of its unsaturated value ΔN. With $I(\omega) = c \cdot \rho(\omega)$, where c

is the light speed, we have

$$\Delta N = \frac{\Delta N_0}{1 + \frac{B_{12} I(\omega)}{C \bar{R}}} = \frac{\Delta N_0}{1 + I/I_s} \qquad (454)$$

where $I_s = C \cdot \bar{R} / B_{12}$ is the saturation intensity and stands for the incidenct intensity that decreases ΔN to $\Delta N_0/2$.

Since the absorption coefficient

$$\alpha = \sigma_{ik} \left[N_i - \frac{g_i}{g_k} N_k \right]$$

$$g_i = g_k = 1 \longrightarrow = \sigma \left[N_i - N_k \right] = \sigma \cdot \Delta N, \qquad (455)$$

we obtain the result that with increasing incident intensity I, the absorption coefficient

$$\alpha = \sigma \cdot \frac{\Delta N_0}{1+S} = \frac{\alpha_0}{1+S} = \frac{\alpha_0}{1 + I/I_s} \qquad (456)$$

where $\alpha_0 = \sigma \cdot \Delta N_0$ is the absorption coefficient for $I = 0$. Eq.(156) shows that when $I \to \infty$, $\alpha \to 0$, i.e., when $I \to \infty$, the population difference $\Delta N = N_1 - N_2 \to 0$, so the absorption and emission rates are close to each other, no attenuation to the incident light — the atom sample becomes transparent.

§7.9. Saturation Broadening of Homogeneous Line Profiles

For population densities N_1 and N_2 on levels E_1 and E_2, when using a radiation field with broad spectral profile and spectral energy density $\rho(\omega)$, the power absorbed per unit volume on the transition $E_1 \to E_2$ by atoms is

$$\frac{dW_{12}}{dt} = \hbar\omega\, B_{12}\rho(\omega)\Delta N$$

$$= \hbar\omega \cdot B_{12}\rho(\omega) \cdot \frac{\Delta N_0}{1+S}$$

$$= \hbar\omega\, \bar{R} \cdot \frac{\Delta N_0}{1+1/S} \quad (457)$$

Since the absorption coefficient $\alpha(\omega)$ of a homogeneously broadened line is Lorentzian, see, e.g., Eq. (381), the induced absorption probability follows a Lorentzian line profile $B_{12}\rho(\omega)\cdot g_L(\omega)$. Therefore, we introduce a frequency-dependent spectral saturation parameter S_ω for transition $E_1 \to E_2$:

$$S_\omega = \frac{B_{12}\,\rho(\omega)}{\langle R\rangle} g_L(\omega). \quad (458)$$

where $\langle R\rangle$ is the mean relaxation rate, and assumed to be independent of ω within the frequency range of the line profile. Recall $g_L = \frac{\gamma/2\pi}{(\omega-\omega_0)^2+(\gamma/2)^2}$, (459)

$$\therefore\ S_\omega = S_0 \frac{(\gamma/2)^2}{(\omega-\omega_0)^2+(\gamma/2)^2} \quad (460)$$ with $S_0 = S_\omega(\omega=\omega_0)$.

$$S_0 = \frac{B_{12}\,\rho(\omega)}{\pi \bar{R}\cdot \gamma/2}. \quad (461)$$

Substitute S_ω given by Eq. (460) as S into Eq. (457), we obtain

$$\frac{dW_{12}}{dt} = \hbar\omega \cdot \bar{R} \cdot \frac{\Delta N_0}{1 + \dfrac{(\omega_0-\omega)^2 + (\gamma/2)^2}{S_0\,(\gamma/2)^2}}$$

$$= \hbar\omega \cdot \bar{R} \cdot \frac{\Delta N_0 \cdot S_0\,(\gamma/2)^2}{(\omega_0-\omega)^2 + (\gamma/2)^2(1+S_0)}$$

$$= \frac{\hbar\omega\,\bar{R}\,\Delta N_0\,S_0\,(\gamma/2)^2}{(\omega_0-\omega)^2 + \left[\gamma\sqrt{1+S_0}/2\right]^2} \qquad (462)$$

$$\therefore\ \frac{dW_{12}}{dt} = \frac{C}{(\omega_0-\omega)^2 + (\gamma_s/2)^2} \qquad (463)$$

Where $C = \hbar\omega\,\bar{R}\,\Delta N_0\,S_0\,(\gamma/2)^2$, (464)

$$\gamma_s = \gamma\sqrt{1+S_0} \qquad (465)$$

Therefore, the FWHW of the saturation-broadened line

$$\delta\omega_s = \gamma_s = \gamma\sqrt{1+S_0} \qquad (466)$$

i.e., $\delta\omega_s$ increases with the saturation parameter S_0

If $S_0 = 1$, then $\delta\omega_s = \sqrt{2}\,\delta\omega_0$ (where $\delta\omega_0$ is the homogeneous broadening width when $\rho \to 0$).

Since the power absorbed per unit volume $\frac{dW_{12}}{dt}$ equals to the intensity decrease per centimeter: $dI/dz = -I\alpha_s$, from Eq.(463) we have:

$$\frac{C}{(\omega_0-\omega)^2+(\gamma_s/2)^2} = +I\alpha_s = c\rho(\omega)\alpha_s \qquad (467)$$

$$\therefore \alpha_s = \frac{C/[c\rho(\omega)]}{(\omega_0-\omega)^2+(\gamma_s/2)^2} = \frac{\hbar\omega\,\Delta N_0\,B_{12}}{\pi c}\cdot\frac{\gamma/2}{(\omega-\omega_0)^2+(\gamma_s/2)^2} \qquad (468)$$

$$\therefore \alpha_s(\omega) = \frac{2\hbar\omega B_{12}\cdot\Delta N_0}{\pi c\gamma}\cdot\frac{(\gamma/2)^2}{(\omega-\omega_0)^2+(\gamma_s/2)^2} \qquad (469)$$

This shows that the saturation decreases the absorption coefficient $\alpha(\omega)$ by the factor $(1+S_\omega)$. The saturation is strongest at the line center, and approaches zero for $|\omega-\omega_0|\to\infty$. This is why the line broadens: absorbed power still increases with increased I,

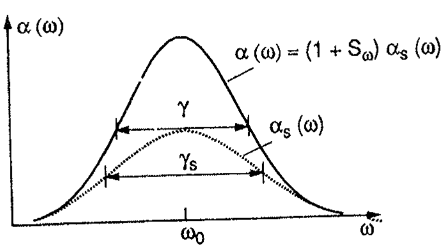
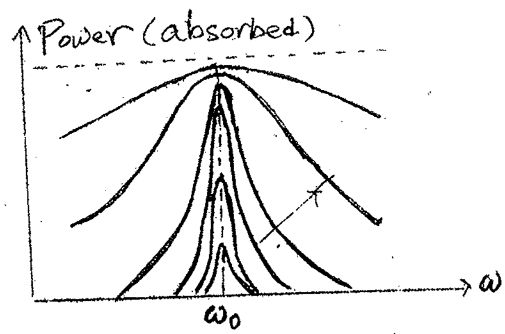


Fig. 3.24. Saturation broadening of a homogeneous line profile



but it increases much slower at the line center than at the wings. thus, the line becomes wider (broadening).

§7.10. Absorption and Dispersion for Doppler-Broadened Spectral Lines

The absorption and dispersion discussed previously under natural linewidth are for atoms at rest. In reality, atoms undergo random thermal motion (in gaseous medium), having Doppler broadening in spectral lines. Thus, we must make corrections to the equations of absorption and dispersion.

Recall the refraction index n is given by

$$n = \sqrt{\varepsilon\mu} = \sqrt{\varepsilon} = \sqrt{1+\chi} \qquad \text{Eqs.(356) and (357)}$$

In the first order approximation

$$n = \sqrt{1+\chi} \simeq 1 + \frac{1}{2}\chi = 1 + \frac{1}{2}\chi' - i\frac{1}{2}\chi'' \qquad (470)$$

Where $\chi = N\chi_e$, Eq.(352) (N is the difference of number densities between the lower and upper levels)

$$= \chi' - i\chi'' \qquad (471)$$

χ is the macroscopic susceptibility.

$\because \; n(\omega) = n'(\omega) - iK(\omega)$ Eq.(360)

$$\therefore \; n'(\omega) = 1 + \frac{1}{2}\chi' \qquad (472)$$

$$K(\omega) = \frac{1}{2}\chi'' \qquad (473)$$

From Eq. (372), the absorption coefficient

$$\alpha(\omega) = 4\pi K(\omega)/\lambda_0 = \frac{2\omega}{c} K(\omega) \qquad (474)$$

Substitute Eq. (473) to Eq. (474), we obtain absorption coefficient

$$\alpha(\omega) = \frac{\omega}{c}\chi'' \qquad (475)$$

dispersion

$$n'(\omega) = 1 + \frac{1}{2}\chi' \qquad (472)$$

* <u>In the natural linewidth case,</u>

From classical model,

$$\chi = \frac{Ne^2}{m\varepsilon_0}\frac{1}{\omega_0^2 - \omega^2 + i\gamma\omega} \qquad \text{Eq.(352)}$$

Considering $|\omega_0 - \omega| << \omega_0$, $\omega_0 + \omega \approx 2\omega_0$, Eq.(352) is simplified to

$$\chi = \frac{Ne^2}{2m\varepsilon_0\omega_0} \cdot \frac{1}{\omega_0 - \omega + i\gamma/2} \qquad (476)$$

With Quantum Mechanics correction, and in a simplified case, i.e., only consider one absorption line,

$$\chi = \frac{Ne^2 f_{ik}}{2m\varepsilon_0\omega_0} \cdot \frac{1}{\omega_0 - \omega + i\gamma/2} \qquad (477)$$

$$= \frac{Ne^2 f_{ik}}{2m\varepsilon_0\omega_0} \cdot \frac{\omega_0 - \omega - i\gamma/2}{(\omega_0 - \omega)^2 + (\gamma/2)^2} \qquad (478)$$

where f_{ik} is the oscillator strength for the absorption line.

Thus, in the nature linewidth case,

absorption coefficient

$$\alpha(\omega) = \frac{\omega}{c}\chi'' = \frac{N e^2 f_{ik}}{2 m \varepsilon_0 c} \cdot \frac{\gamma/2}{(\omega_0-\omega)^2 + (\gamma/2)^2} \qquad (479)$$

dispersion

$$n'(\omega) = 1 + \frac{1}{2}\chi'$$

$$= 1 + \frac{N e^2 f_{ik}}{4 m \varepsilon_0 \omega_0} \cdot \frac{\omega_0 - \omega}{(\omega_0-\omega)^2 + (\gamma/2)^2} \qquad (480)$$

Define normalized Lorentzian lineshape:

$$g_L(\omega) = \frac{\gamma/2\pi}{(\omega_0-\omega)^2 + (\gamma/2)^2} \qquad (481)$$

$$\therefore \quad \alpha^L(\omega) = \frac{N e^2 f_{ik}}{2 m \varepsilon_0 c} \pi \, g_L(\omega) \qquad (482)$$

* In the Doppler linewidth case,

for atoms moving with velocity component v along the laser beam $\vec{k}$, the ω in Eq. (477) should be changed:

$$\omega \longrightarrow \omega - kv \qquad (483)$$

Since v has Maxwellian distribution $W(v)$, the macroscopic susceptibility χ becomes

$$\chi = \frac{Ne^2 f_{ik}}{2m\varepsilon_0\omega_0}\int_{-\infty}^{+\infty}\frac{W(v)}{\omega_0-\omega+kv+i\gamma/2}\,dv, \qquad (484)$$

where $W(v)$ is normalized Gaussian shape:

$$W(v) = \frac{1}{\sqrt{\pi}\,u}\exp\left[-\frac{v^2}{u^2}\right] \qquad (485)$$

where $u^2 \equiv \frac{2k_BT}{M}$ (here M is the atomic mass) ← (186)

Eq. (484) can be written into a standard format:

$$\chi = \frac{Ne^2 f_{ik}}{2m\varepsilon_0\omega_0 ku}\int_{-\infty}^{+\infty}\frac{ku\,W(v)}{\omega_0-\omega+kv+i\gamma/2}\,dv$$

electron mass

$$= \frac{Ne^2 f_{ik}}{2m\varepsilon_0\omega_0 ku}\,Z(\omega), \qquad (487)$$

where $$Z(\omega) = \int_{-\infty}^{+\infty}\frac{ku\,W(v)}{\omega_0-\omega+kv+i\gamma/2}\,dv \qquad (488)$$

Let $\xi \equiv (\omega-\omega_0-i\gamma/2)/ku$, then $Z(\omega)$ can be converted to plasma dispersion function:

$$Z(\xi) = -2ie^{-\xi^2}\int_{-\infty}^{i\xi} e^{-t^2}\,dt \qquad (489)$$

To simplify, take the limit of Doppler width, i.e., let $\gamma \to 0$ (natural linewidth disappears $\longrightarrow$ single line).

$$\therefore\ \xi \to (\omega-\omega_0)/ku \equiv \eta \qquad (490)$$

$$\therefore Z(\xi) \to Z(\eta) = -2ie^{-\eta^2}\left[\int_{-\infty}^{0} e^{-t^2}dt + \int_0^{i\eta} e^{-t^2}dt\right]$$

$$= -i\sqrt{\pi}\, e^{-\eta^2} + 2e^{-\eta^2}\int_0^{\eta} e^{t^2}dt \qquad (491)$$

$(t \to it)$

$$\therefore \chi' = \frac{Ne^2 f_{ik}}{2m\varepsilon_0\omega_0 ku}\left[2e^{-\eta^2}\int_0^{\eta} e^{t^2}dt\right] \qquad (492)$$

$$\chi'' = \frac{Ne^2 f_{ik}}{2m\varepsilon_0\omega_0 ku}\sqrt{\pi}\, e^{-\eta^2} \qquad (493)$$

Thus, in Doppler linewidth, the absorption coefficient

$$\alpha(\omega) = \frac{\omega}{c}\chi''$$

$$= \frac{Ne^2 f_{ik}}{2m\varepsilon_0 c\; ku}\sqrt{\pi}\, e^{-\frac{(\omega-\omega_0)^2}{(ku)^2}} \qquad (494)$$

$$\boxed{k = \frac{2\pi}{\lambda},\ u = \sqrt{\frac{2k_BT}{M}}} \qquad \sigma_D = \sqrt{\frac{k_BT}{M\lambda^2}}$$ (in ν unit, not ω)

$$ku = \frac{2\pi}{\lambda}\sqrt{\frac{2k_BT}{M}} = \sqrt{2}\cdot 2\pi\cdot\sqrt{\frac{k_BT}{M\lambda^2}}$$

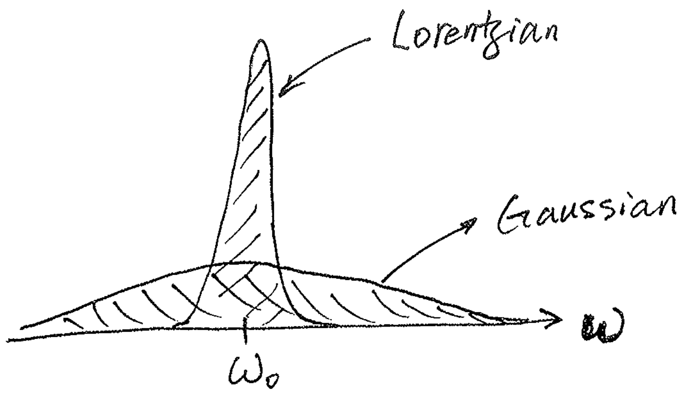
$$= \sqrt{2}\cdot 2\pi\cdot\sigma_D \qquad (495)$$

$$\therefore \alpha(\omega) = \frac{Ne^2 f_{ik}}{2m\varepsilon_0 c}\cdot\pi\cdot\frac{1}{\sqrt{2\pi}\,(2\pi\cdot\sigma_D)}\, e^{-\frac{(\omega-\omega_0)^2}{2(2\pi\sigma_D)^2}} \qquad (496)$$

Define normalized Gaussian lineshape

$$g_D(\omega) = \frac{1}{\sqrt{2\pi}\,(2\pi\cdot\sigma_D)}\, e^{-\frac{(\omega-\omega_0)^2}{2(2\pi\sigma_D)^2}} \qquad (497)$$

$$\therefore\ \alpha^{D}(\omega) = \frac{N e^2 f_{ik}}{2 m \varepsilon_0 c} \cdot \pi \cdot g_D(\omega) \qquad (498).$$



Compare Eq. (482) and Eq. (498).

$$\int \alpha^{L}(\omega)\, d\omega = \frac{N e^2 f_{ik}\, \pi}{2 m \varepsilon_0 c} \int g_L(\omega)\, d\omega = \frac{N e^2 f_{ik}\, \pi}{2 m \varepsilon_0 c}$$

$$\int \alpha^{D}(\omega)\, d\omega = \frac{N e^2 f_{ik}\, \pi}{2 m \varepsilon_0 c} \int g_D(\omega)\, d\omega = \frac{N e^2 f_{ik}\, \pi}{2 m \varepsilon_0 c}.$$

i.e., the areas below the absorption coefficient curve are equal for the same amount of atoms N for Lorentzian or Gaussian shapes.

But the natural linewidth (Lorentzian) is much narrower than the Doppler linewidth (Gaussian), the peak absorption coefficient $\alpha(\omega_0)$ is much larger in the Lorentzian case than the Doppler case.

Let us calculate the peak absorption coefficient:

$$\alpha^L(\omega_0) = \frac{N e^2 f_{ik}}{2 m \varepsilon_0 c\, (\gamma/2)}$$

$$\delta\omega_L = \gamma \quad \underline{\underline{=}} \quad \frac{N e^2 f_{ik}}{m \varepsilon_0 c \cdot \delta\omega_L}$$

$$= \frac{e^2}{m \varepsilon_0 c} \cdot \frac{N f_{ik}}{\delta\omega_L}$$

$$= \frac{(1.601 \times 10^{-19})^2}{9.109 \times 10^{-31} \times 8.854 \times 10^{-12} \times 299792458} \cdot \frac{N f_{ik}}{\delta\omega_L}$$

$$= 1.06 \times 10^{-5} \frac{N f_{ik}}{\delta\omega_L} \qquad (499)$$

$$\alpha^D(\omega_0) = \frac{N e^2 f_{ik}}{2 m \varepsilon_0 c} \cdot \frac{\pi}{\sqrt{2\pi}\,(2\pi \sigma_D)}$$

$$\delta\omega_D = \sqrt{8 \ln 2}\,(2\pi\sigma_D) \quad \underline{\underline{=}} \quad \frac{N e^2 f_{ik} \cdot \pi \cdot \sqrt{8 \ln 2}}{2 m \varepsilon_0 c \cdot \sqrt{2\pi} \cdot \delta\omega_D}$$

$$= \frac{e^2 \pi \sqrt{8 \ln 2}}{2 m \varepsilon_0 c \sqrt{2\pi}} \cdot \frac{N f_{ik}}{\delta\omega_D}$$

$$= \frac{(1.601 \times 10^{-19})^2 \times \pi \times \sqrt{8 \ln 2}}{2 \times 9.109 \times 10^{-31} \times 8.854 \times 10^{-12} \times 299792458 \sqrt{2\pi}} \cdot \frac{N f_{ik}}{\delta\omega_D}$$

$$= 1.564 \times 10^{-5} \frac{N f_{ik}}{\delta\omega_D}. \qquad (500).$$