4. Magnetic Dipole Transition and Electric Quadrapole Transition

$$e^{i\vec{k}\cdot\hat{\vec{r}}} = 1 + i\vec{k}\cdot\hat{r} - \frac{(\vec{k}\cdot\hat{\vec{r}})^2}{2!} + \cdots$$

When we consider the 2nd term $i\vec{k}\cdot\hat{\vec{r}}$ and the $\frac{e}{\mu}\hat{\vec{S}_i}\cdot\hat{\vec{B}}$ term in equation (198), we will obtain the interaction between the radiation field and the atomic magnetic dipole moment, and the atomic quadrapole moment electric.

Since the transition probability $\propto$ the square of matrix element,

$$dW_{sp}^{(M_1, E_2)} \propto dW_{sp}^{(E_1)}\cdot(kr)^2$$

$$\therefore \frac{dW_{sp}^{(M_1,E_2)}}{dW_{sp}^{(E_1)}} \propto (kr)^2 = \left(\frac{2\pi a_0}{\lambda}\right)^2 \approx 10^{-6}. \qquad (274)$$

$\therefore$ M_1, E_2 transition probability is much smaller than E_1.

Considering $i\vec{k}\cdot\hat{\vec{r}}$ term,

$$\langle b|\hat{\vec{e}}_{k\lambda}\cdot\hat{\vec{P}}e^{\pm i\vec{k}\cdot\hat{\vec{r}}}|a\rangle \longrightarrow \langle b|\pm i\hat{\vec{e}}_{k\lambda}\cdot\hat{\vec{P}}\vec{k}\cdot\hat{\vec{r}}|a\rangle$$

$$i\hat{\vec{e}}_{k\lambda}\cdot\hat{\vec{P}}\vec{k}\cdot\hat{\vec{r}} = i\hat{\vec{e}}_{k\lambda}\cdot\hat{\vec{P}}\hat{\vec{r}}\cdot\vec{k}$$

$$= \frac{1}{2}i\hat{\vec{e}}_{k\lambda}\cdot(\hat{\vec{P}}\hat{\vec{r}} - \hat{\vec{r}}\hat{\vec{P}})\cdot\vec{k} + \frac{1}{2}i\hat{\vec{e}}_{k\lambda}\cdot(\hat{\vec{P}}\hat{\vec{r}} + \hat{\vec{r}}\hat{\vec{P}})\cdot\vec{k}$$

$$\equiv \hat{O}_A + \hat{O}_S \qquad (275)$$

$\hat{O}_A$: Asymmetry operator $\downarrow$ M_1 Transition

$\hat{O}_S$: Symmetry operator $\downarrow$ E_2 Transition.

(1) Magnetic Dipole (M_1) Transition

Because $(\hat{\vec{A}} \times \hat{\vec{B}}) \cdot (\hat{\vec{C}} \times \hat{\vec{D}}) = (\hat{\vec{A}} \cdot \hat{\vec{C}})(\hat{\vec{B}} \cdot \hat{\vec{D}}) - (\hat{\vec{A}} \cdot \hat{\vec{D}})(\hat{\vec{B}} \cdot \hat{\vec{C}})$

$$\therefore \hat{O}_A = \frac{i}{2} \hat{\vec{e}}_{k\lambda} \cdot (\hat{\vec{p}}\hat{\vec{r}} - \hat{\vec{r}}\hat{\vec{p}}) \cdot \hat{\vec{k}}$$

$$= \frac{i}{2} [(\hat{\vec{e}}_{k\lambda} \cdot \hat{\vec{p}})(\hat{\vec{r}} \cdot \hat{\vec{k}}) - (\hat{\vec{e}}_{k\lambda} \cdot \hat{\vec{r}})(\hat{\vec{p}} \cdot \hat{\vec{k}})]$$

$$= \frac{i}{2} (\hat{\vec{k}} \times \hat{\vec{e}}_{k\lambda}) \cdot (\hat{\vec{r}} \times \hat{\vec{p}}) \qquad \hat{\vec{k}} = \hat{\vec{k}}_0 \frac{\omega}{c} = \hat{\vec{k}}_0 \frac{2\pi}{\lambda}$$

$$= \frac{i\omega}{2c} (\hat{\vec{k}}_0 \times \hat{\vec{e}}_{k\lambda}) \cdot \hat{\vec{L}} \qquad \hat{\vec{r}} \times \hat{\vec{p}} = \hat{\vec{L}}$$

$$= -i \frac{\mu\omega}{ec} (\hat{\vec{k}}_0 \times \hat{\vec{e}}_{k\lambda}) \cdot \hat{\vec{\mu}}_L \qquad (276)$$

where $$\hat{\vec{\mu}}_L = -\frac{e\hbar}{2m_e}\sqrt{L(L+1)} = -\frac{e}{2m_e} g_L \hat{\vec{L}} = -\frac{e}{2\mu} g_L \hat{\vec{L}} \qquad (277)$$

is the orbital magnetic moment of the electron. $(g_L = 1)$

$\hat{\vec{k}}_0 \times \hat{\vec{e}}_{k\lambda}$ is the unit vector along the magnetic field $\hat{\vec{B}}$.

From Eq. (203), we obtain the interaction Hamiltonian operator:

$$\hat{H}_{int} = \frac{e}{\mu} \sqrt{\frac{\hbar}{2\varepsilon_0 V\omega}} \hat{O}_A (\hat{a}_{k\lambda} - \hat{a}_{k\lambda}^+)$$

$$= -\frac{i\omega}{c} \sqrt{\frac{\hbar}{2\varepsilon_0 V\omega}} (\hat{\vec{k}}_0 \times \hat{\vec{e}}_{k\lambda}) \cdot \hat{\vec{\mu}}_L (a_{k\lambda} - \hat{a}_{k\lambda}^+)$$

Eq.(169)
$$= - \hat{\vec{B}}(\hat{\vec{r}}) \cdot \hat{\vec{\mu}}_L \qquad (278)$$

The $\frac{e}{\mu} \hat{\vec{S}}_i \cdot \hat{\vec{B}}$ term gives the interaction between electron spin magnetic moment and the radiation field. Through some algorithm, we obtain the Hamoltonian operator for M_1 transition:

$$\hat{H}_{int} = - \hat{\vec{B}}(\hat{\vec{r}}) \cdot \hat{\vec{\mu}} \qquad (279)$$

Where $\hat{\vec{\mu}}$ is the total magnetic dipole moment, including the electron orbital, electron spin, and nuclear spin contributions.

This is very similar to the electric dipole interaction:

$$\hat{H}_{int} = -\hat{\vec{E}}(\hat{\vec{r}}) \cdot \hat{\vec{d}} \qquad (280)$$

Therefore, we can use similar derivation procedure to obtain the transition probability per unit time per solid angle:

$$dW_{sp} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{\omega^3}{2\pi\hbar c^5} \left| (\hat{\vec{k}}_0 \times \hat{\vec{e}}_{k\lambda}) \cdot \langle b | \hat{\vec{\mu}} | a \rangle \right|^2 d\Omega \qquad (281)$$

$$dW_{st} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{4\pi^2}{\hbar^2 c^3} I_{k\lambda} \left| (\hat{\vec{k}}_0 \times \hat{\vec{e}}_{k\lambda}) \cdot \langle b | \hat{\vec{\mu}} | a \rangle \right|^2 d\Omega \qquad (282)$$

When not considering hyperfine structure, $\langle b | \hat{\vec{\mu}} | a \rangle$ can be written as $\langle \alpha_b L_b S_b J_b M_b | \hat{\vec{\mu}} | \alpha_a L_a S_a J_a M_a \rangle$

$$= (-1)^{J_b - M_b} \begin{pmatrix} J_b & 1 & J_a \\ -M_b & q & M_a \end{pmatrix} \langle \alpha_b L_b S_b J_b \| \hat{\vec{\mu}} \| \alpha_a L_a S_a J_a \rangle \qquad (283)$$

where $\langle \alpha_b L_b S_b J_b \| \hat{\vec{\mu}} \| \alpha_a L_a S_a J_a \rangle = -\frac{e}{2\mu} \langle \alpha_b L_b S_b J_b \| \hat{\vec{L}} + 2\hat{\vec{S}} \| \alpha_a L_a S_a J_a \rangle$ (284)

The selection rules are

$$\Delta\alpha = 0 \ (\text{i.e., } \Delta n = 0), \quad \Delta L = 0, \quad \Delta S = 0$$

$$\Delta J = 0, \pm 1, \quad J_a + J_b \geq 1; \quad \Delta M = 0, \pm 1; \text{ no parity change.} \qquad (285)$$

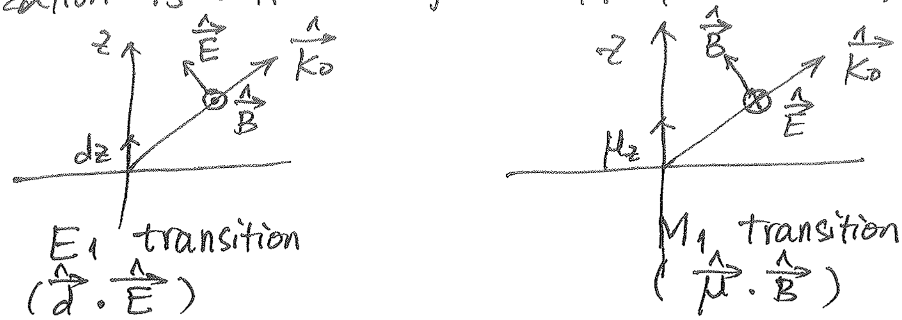
When hyperfine structure is involved, additional selection rules are

$$\Delta F = 0, \pm 1, \quad F_a + F_b \geq 1 \ (\text{i.e., } F = 0 \not\leftrightarrow F = 0)$$

$$\Delta m_F = 0, \pm 1. \qquad (286)$$

M_1 transitions only occur between fine structure or hyperfine structure (because $\Delta n = 0$), resulting in low transition frequency, ~ radio- or microwave frequency, and A_{ki} is small.

The angular distribution of M1 radiation is similar to E1 transition, but the polarization is different from E1. For $\Delta M=0$, see the figure below:



For E1: $\vec{E}$ is in the plane of $(\hat{\vec{k}}_0, d_z)$

For M1: $\vec{B}$ is in the plane of $(\hat{\vec{k}}_0, \mu_z)$, but $\vec{E}$ is perpendicular to this plane.

From $E_1 \to M$, the following transformation:

$$\hat{\vec{d}} \to \hat{\vec{\mu}}, \quad \hat{\vec{E}} \to \hat{\vec{B}}, \quad \hat{\vec{B}} \to -\hat{\vec{E}} \qquad (287).$$

After integrating through all possible directions,

$$W_{sp}(\alpha_a J_a \to \alpha_b J_b) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{4\omega^3}{3\hbar c^5}\cdot\frac{1}{2J_a+1}\left|\langle\alpha_b J_b\|\hat{\vec{\mu}}\|\alpha_a J_a\rangle\right|^2 \qquad (288)$$

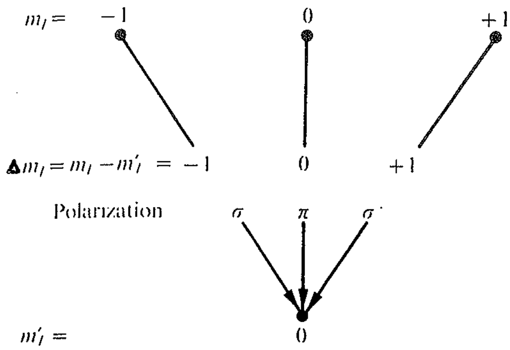
$$W_{st}(\alpha_a J_a \to \alpha_b J_b) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{4\pi^2}{3\hbar^2 c^2}\rho(\omega)\frac{1}{2J_a+1}\left|\langle\alpha_b J_b\|\hat{\vec{\mu}}\|\alpha_a J_a\rangle\right|^2 \qquad (289)$$

Polarization of Electric Dipole Transitions and Magnetic Dipole Transitions

(1) Polarization of Electric Dipole Transitions (E_1 Transitions)

$\Delta M = 0$, π – polarization (linear polarization)

$\Delta M = \pm 1$, σ – polarization (circular polarization)



Selection rules for magnetic quantum number m_l and polarization of electric dipole radiation (E_1) observed in the direction $\theta = 0$.

(2) Polarization of Magnetic Dipole Transitions (M_1 transitions)

$\Delta M = \pm 1$, π – polarization (linear polarization for $\vec{E}$)

$\Delta M = 0$, σ – polarization (circular polarization for $\vec{E}$)

Selection rules for magnetic dipole transitions between hyperfine levels belonging to states with the same electronic angular momentum J (M_1) Transition

Static magnetic field $\underline{B}$	π-polarization ($\underline{B}_1 \perp \underline{B}$)	σ-polarization ($\underline{B}_1 \parallel \underline{B}$)
Weak	$\Delta M_F = \pm 1$	$\Delta M_F = 0$
Weak	$\Delta F = 0, \pm 1$	$\Delta F = \pm 1$
Strong	$\Delta M_J = \pm 1$	$\Delta M_J = 0$
Strong	$\Delta M_I = 0$	$\Delta M_I = 0$

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(2) Electric Quadrapole Transitions (E_2)

$$\hat{O}_s = \frac{i}{2}\hat{e}_{k\lambda}\cdot(\hat{\vec{p}}\hat{\vec{r}}+\hat{\vec{r}}\hat{\vec{p}})\cdot\hat{\vec{k}} \qquad (290)$$

$\because$ $\hat{\vec{p}} = \frac{i\mu}{\hbar}[\hat{H}_a, \hat{\vec{r}}]$

$$\therefore \hat{O}_s = -\frac{\mu}{2\hbar}\hat{\vec{e}}_{k\lambda}\cdot\{[\hat{H}_a,\hat{\vec{r}}]\hat{\vec{r}}+\hat{\vec{r}}[\hat{H}_a,\hat{\vec{r}}]\}\cdot\hat{\vec{k}}$$

$$= -\frac{\mu}{2\hbar}\hat{\vec{e}}_{k\lambda}\cdot[\hat{H}_a, \hat{\vec{r}}\hat{\vec{r}}]\cdot\hat{\vec{k}} \qquad (291)$$

$$\therefore \langle b|\hat{O}_s|a\rangle = -\frac{\mu}{2\hbar}(E_b-E_a)\hat{\vec{e}}_{k\lambda}\cdot\langle b|\hat{\vec{r}}\hat{\vec{r}}|a\rangle\cdot\hat{\vec{k}}$$

$$= -\frac{\mu\omega}{2}\hat{\vec{e}}_{k\lambda}\cdot\langle b|\hat{\vec{r}}\hat{\vec{r}}|a\rangle\cdot\hat{\vec{k}} \qquad (292)$$

Since $\hat{\vec{e}}_{k\lambda}\cdot\langle b|r^2\delta_{ij}|a\rangle\cdot\hat{\vec{k}} = \langle b|r^2\delta_{ij}|a\rangle\,\hat{\vec{e}}_{k\lambda}\cdot\hat{\vec{k}} = 0 \qquad (293)$

then in Eq. (292), we can replace $\hat{\vec{r}}\hat{\vec{r}}$ with $\hat{\vec{r}}\hat{\vec{r}} - r^2\delta_{ij}/3$

Define $\quad \hat{Q} \equiv \hat{\vec{r}}\hat{\vec{r}} - \frac{1}{3}r^2\delta_{ij} \qquad (294),$

then $\hat{Q}$ is the electric quadrapole of the atom.

$$\langle b|\hat{Q}|a\rangle = -\frac{\mu\omega}{2}\hat{\vec{e}}_{k\lambda}\cdot\langle b|\hat{Q}|a\rangle\cdot\hat{\vec{k}}$$

$$= -\frac{\mu\omega^2}{2c}\hat{\vec{e}}_k\cdot\langle b|\hat{Q}|a\rangle\cdot\hat{\vec{k}}_0 \qquad (295).$$

$\hat{\vec{k}}_0$ → unit vector

The interaction Hamiltonian operator is

$$\hat{H}_{int} = \frac{e}{\mu}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega}}\,\hat{O}_s\,(\hat{a}_{k\lambda}-\hat{a}^+_{k\lambda})$$

$$= -\frac{e\omega}{2c}\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\,(\hat{\vec{e}}_{k\lambda}\cdot\hat{Q}\cdot\hat{\vec{k}}_0)\,(\hat{a}_{k\lambda}-\hat{a}^+_{k\lambda}) \qquad (296)$$

Regarding Eq. (296) as perturbation, we can use time-dependent perturbation theory to derive the transition probability, angular distribution, and polarization of E_2 transition.

Here we only analyze the selection rules.

$$\langle b | \hat{\vec{e}}_{k\lambda} \cdot \hat{Q} \cdot \hat{\vec{k}}_0 | a \rangle = \hat{\vec{e}}_{k\lambda} \cdot \langle b | Q | a \rangle \cdot \hat{\vec{k}}_0$$

$$\langle \alpha_b L_b S_b J_b M_b | Q_q^{(2)} | \alpha_a L_a S_a J_a M_a \rangle$$

$$= (-1)^{J_b - M_b} \begin{pmatrix} J_b & 2 & J_a \\ -M_b & q & M_a \end{pmatrix} \langle \alpha_b L_b S_b J_b \| Q^{(2)} \| \alpha_a L_a S_a J_a \rangle$$

The selection rules are

$\Delta L = 0, \pm 1, \pm 2, \quad L_a + L_b \geq 2; \quad \Delta S = 0$

$\Delta J = 0, \pm 1, \pm 2, \quad J_a + J_b \geq 2, \quad \Delta M = 0, \pm 1, \pm 2$

No parity change.

For hyperfine structure, add : $\Delta I = 0$.

* Example for E_2 and M_1 transitions: Oxygen atom OI.

OI ground state electron configuration: $1s^2 2s^2 2p^4$.

p^4 configuration has very similar atomic state terms as p^2.

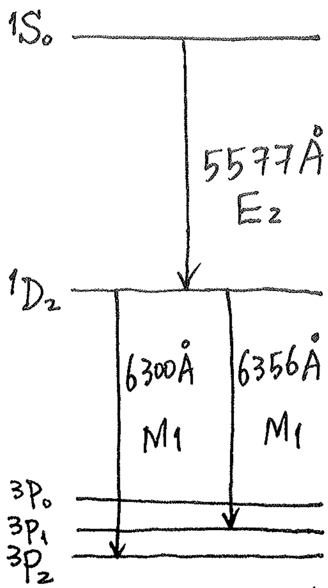
$$p^2: \quad \left.\begin{array}{l} s_1 = \frac{1}{2},\ s_2 = \frac{1}{2} \Rightarrow S = 0,\ 1 \\ l_1 = 1,\ l_2 = 1 \Rightarrow L = 0, 1, 2 \end{array}\right\} {}^1S_0,\ {}^1D_2,\ {}^3P_{0,1,2} \text{ for equivalent electrons.}$$

According to Hund's rule, the ground state term is the term of maximum multiplicity, i.e., the term ^3P.

As the p shell is less than half filled in this p^2 configuration, the levels $J=0,1,2$ are arranged in normal order, i.e., the level $J=0$ is the lowest level. Thus, $^3P_0$ is the ground state for p^2.

As for p^4, the p shell is more than half filled, the level $J=0,1,2$ are arranged in reverse order, i.e., $J=2$ is the lowest. Thus, $^3P_2$ is the ground state for p^4 configuration.

From above analysis, we know the ground state of OI is $2\,{}^3P_2$.



There is no E_1 transition between $^1S_0$ and $^1D_2$ to ^3P states, because $\Delta S \neq 0$. Therefore, $^1S_0$ and $^1D_2$ are both metastable states.

However, there are "famous" transitions between them due to E_2 and M_1 transitions.

From $^1S_0 \rightarrow {}^1D_2$, the famous green line 557.7nm Airglow OI 5577Å line is originating here! Since $\Delta L = 2$, $\Delta J = 2$, this is an E_2 transition.

As from $^1D_2 \rightarrow {}^3P$, because $^1D_2$ actually contains

Atomic Oxygen (OI) Ground States

$^3P_2$ wave function, it is M_1 transitions:

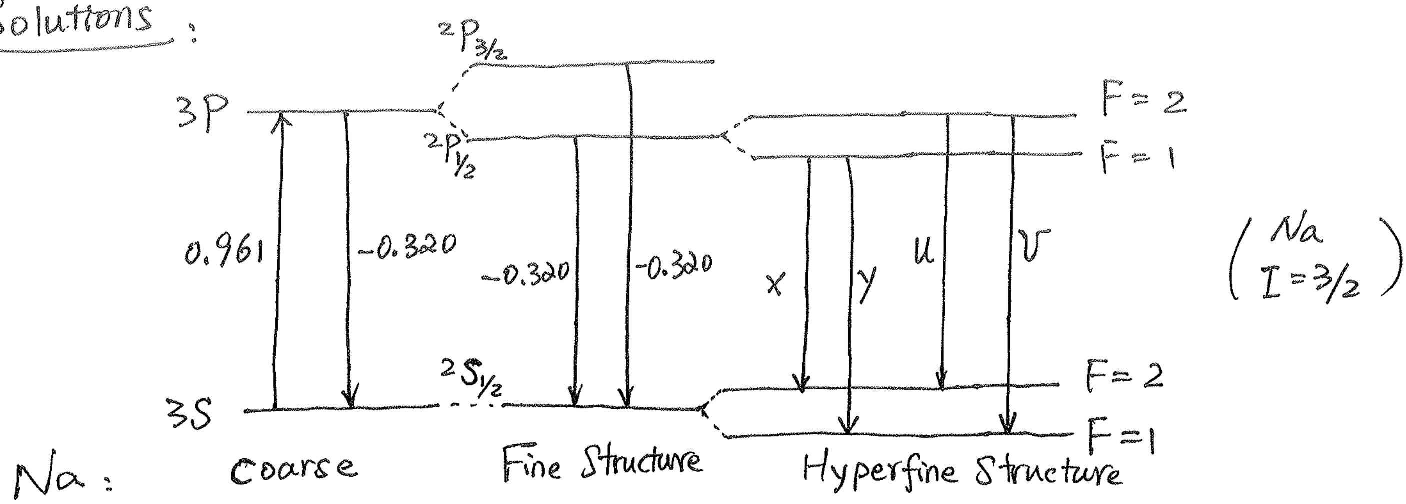
$$^3P_2 \rightarrow {}^3P_2 \quad \text{and} \quad {}^3P_2 \rightarrow {}^3P_1$$

Corresponding to the famous double red lines 6300Å and 6356Å.

Example #1 for Radiative Transition Probability Computation

Problem: Given that Na $f(3S \to 3P) = 0.961$, find the transition probability of the four hyperfine components of D_1 line.

Solutions:



Na oscillator strength $f(3S \to 3P) = 0.961$.

From $-g_{La} f(La \to Lb) = g_{Lb} f(Lb \to La)$, we get

$$f(3P \to 3S) = -\frac{g_{3S}}{g_{3P}} f(3S \to 3P)$$

$$= -\frac{2 \times 0 + 1}{2 \times 1 + 1} \times 0.961 = -\frac{1}{3} \times 0.961$$

$$= -0.320$$

From $f(La \to Lb) = \sum_{Jb} f(Ja \to Jb)$, we obtain

$$f(3P \to 3S) = f(3^2P_{1/2} \to 3^2S_{1/2}) = f(3^2P_{3/2} \to 3^2S_{1/2})$$

$$\therefore \ f(3^2P_{1/2} \to 3^2S_{1/2}) = f(3^2P_{3/2} \to 3^2S_{1/2}) = -0.320.$$

D_1 line is the transition from $3^2P_{1/2} \to 3^2S_{1/2}$. Due to the nuclear spin of Na $I = 3/2$, there are hyperfine structures in $3^2P_{1/2}$ and $3^2S_{1/2}$. Accordig to the selection rules, $\Delta F = 0, \pm 1$, so there are 4 hyperfine components in the D_1 spectral line, as illustrated in the above figure.

Let $x = f(3^2P_{1/2}, F=1 \rightarrow 3^2S_{1/2}, F=2)$

$y = f(3^2P_{1/2}, F=1 \rightarrow 3^2S_{1/2}, F=1)$

$u = f(3^2P_{1/2}, F=2 \rightarrow 3^2S_{1/2}, F=2)$

$v = f(3^2P_{1/2}, F=2 \rightarrow 3^2S_{1/2}, F=1)$

As the frequency difference among these 4 hyperfine lines are small, we treat them as the same frequency $\omega = 2\pi\nu = 2\pi\frac{c}{\lambda}$.

Since $f(J_a \rightarrow J_b) = \sum_{F_b} f(F_a \rightarrow F_b)$, we have

$$f(3^2P_{1/2} \rightarrow 3^2S_{1/2}) = x + y = u + v = -0.320.$$

$$f(F_a \rightarrow F_b) = \frac{2\mu\omega}{3\hbar e^2} \cdot \frac{1}{2F_a+1} |\langle \alpha_b J_b I_b F_b \| \hat{\vec{d}} \| \alpha_a J_a I_a F_a \rangle|^2.$$

From Group theory,

$$\langle \alpha_b J_b I_b F_b \| \hat{\vec{d}} \| \alpha_a J_a I_a F_a \rangle$$

$$= (-1)^{J_a+I_a+F_a+1} \delta_{I_b I_a} \sqrt{(2F_a+1)(2F_b+1)} \begin{Bmatrix} F_b & 1 & F_a \\ J_a & I_a & J_b \end{Bmatrix} \langle \alpha_b J_b \| \hat{\vec{d}} \| \alpha_a J_a \rangle$$

$$\therefore f(F_a \rightarrow F_b) = \frac{2\mu\omega}{3\hbar e^2} \cdot \frac{\delta^2_{I_b I_a}}{2F_a+1} (2F_a+1)(2F_b+1) \begin{Bmatrix} F_b & 1 & F_a \\ J_a & I_a & J_b \end{Bmatrix}^2 |\langle \alpha_b J_b \| \hat{\vec{d}} \| \alpha_a J_a \rangle|^2$$

Lect $c = \frac{2\mu\omega}{3\hbar e^2} |\langle \alpha_b J_b \| \hat{\vec{d}} \| \alpha_a J_a \rangle|^2$

$$\therefore f(F_a \rightarrow F_b) = c \cdot \delta^2_{I_b I_a} \cdot (2F_b+1) \begin{Bmatrix} F_b & 1 & F_a \\ J_a & I_a & J_b \end{Bmatrix}^2$$

For x component, $J_a = \frac{1}{2}$, $I_a = \frac{3}{2}$, $F_a = 1$, $J_b = \frac{1}{2}$, $I_b = \frac{3}{2}$, $F_b = 2$

$$\therefore x = c \cdot 1 \cdot (2\times 2+1) \begin{Bmatrix} 2 & 1 & 1 \\ \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \end{Bmatrix}^2 = c \times 5 \times \left(\frac{1}{2\sqrt{3}}\right)^2 = \frac{5}{12}c.$$

Similarly,

$$y = c \cdot 1 \cdot (2\times 1+1) \begin{Bmatrix} 1 & 1 & 1 \\ \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \end{Bmatrix}^2 = c \times 3 \times \left(-\frac{1}{6}\right)^2 = \frac{1}{12}c.$$

$$u = c \cdot 1 \cdot (2\times2+1)\begin{Bmatrix} 2 & 1 & 2 \\ \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \end{Bmatrix} = c\times5\times\left(\frac{1}{2\sqrt{5}}\right)^2 = \frac{1}{4}c = \frac{3}{12}c$$

$$v = c \cdot 1 \cdot (2\times1+1)\begin{Bmatrix} 1 & 1 & 2 \\ \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \end{Bmatrix} = c\times3\times\left(\frac{1}{2\sqrt{3}}\right)^2 = \frac{1}{4}c = \frac{3}{12}c$$

$\therefore\ x : y : u : v = 5 : 1 : 3 : 3$

Substitute this into the above equation $x+y = u+v = -0.320$

$$\left\{\begin{matrix} x+y = -0.320 \\ u+v = -0.320 \end{matrix}\right\} \Rightarrow \quad \begin{aligned} x &= \tfrac{5}{6}\times(-0.320) = -0.267 \\ y &= \tfrac{1}{6}\times(-0.320) = -0.053 \\ u &= \tfrac{1}{2}\times(-0.320) = -0.160 \\ v &= \tfrac{1}{2}\times(-0.320) = -0.160 \end{aligned}$$

The spontaneous transition probability per unit time is given by

$$W_{sp}(\alpha_a J_a I_a F_a \to \alpha_b J_b I_b F_b) = \frac{e^2\omega^2}{2\pi\varepsilon_0 \mu c^3}[-f(F_a\to F_b)] \quad (\text{in SI unit})$$

$$= \frac{-2\pi e^2}{\varepsilon_0 \mu c \lambda^2} f(F_a\to F_b)$$

$$= \frac{-2\pi\times(1.6\times10^{-19})^2 \ f(F_a\to F_b)}{8.854\times10^{-12}\times9.1\times10^{-31}\times3\times10^{8}\times(589.6\times10^{-9})^2}$$

$$= 6.65\times10^{-5}\frac{[-f(F_a\to F_b)]}{\lambda^2}$$

$$= -1.914\times10^{8} f(F_a\to F_b) = 1.914\times10^{8}\,|f(F_a\to F_b)|$$

$\therefore\ W_{sp}(3^2P_{1/2}, F=1 \to 3^2S_{1/2}, F=2) = 1.914\times10^8\times|x| = 1.914\times10^8\times0.267 = 0.511\times10^8 s^{-1}$

$W_{sp}(3^2P_{1/2}, F=1 \to 3^2S_{1/2}, F=1) = 1.914\times10^8\times|y| = 1.914\times10^8\times0.053 = 0.101\times10^8 s^{-1}$

$W_{sp}(3^2P_{1/2}, F=2 \to 3^2S_{1/2}, F=2) = 1.914\times10^8\times|u| = 0.306\times10^8 s^{-1}$

$W_{sp}(3^2P_{1/2}, F=2 \to 3^2S_{1/2}, F=1) = 1.914\times10^8\times|v| = 0.306\times10^8 s^{-1}$.

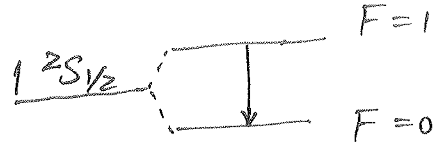
$\therefore$ The ratio of transition probability is $5:1:3:3$.

The ratio of spectral intensity is $5\times(2F_{a_1}+1) : 1\times(2F_{a_1}+1) : 3\times(2F_{a_2}+1) : 3\times(2F_{a_2}+1) = 5\times3 : 1\times3 : 3\times5 : 3\times5 = 5:1:5:5$.

Example #2 for Radiative Transition Probability Calculation

<u>Problem</u>: Find the transition probability of $F=1 \to 0$ transition (spontaneous emission) within the ground state $^2S_{1/2}$ of hydrogen.

<u>Solutions</u>: H:



$l=0,\ s=1/2,\ j=\frac{1}{2}$

$I=\frac{1}{2}$, $\therefore F=1, 0$.

The transition between $F=1 \leftrightarrow 0$ of the hyperfine structure of the ground state is a magnetic dipole transition (M_1).

The transition probability per unit time per unit solid angle is

$$dW_{sp}(F=1\to 0) = \frac{1}{4\pi\varepsilon_0}\frac{\omega^3}{2\pi\hbar c^5}\left|(\vec{k}_0\times\vec{e}_{k\lambda})\cdot\langle\alpha_b J_b I_b F_b \| \hat{\vec{\mu}} \| \alpha_a J_a I_a F_a\rangle\right|^2 \cdot d\Omega$$

Integrating over all possible directions, i.e., 4π solid angle, the total spontaneous transition probability per unit time is

$$W_{sp}(F=1\to 0) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{4\omega^3}{3\hbar c^5}\cdot\frac{1}{2F_a+1}\left|\langle\alpha_b J_b I_b F_b \| \hat{\vec{\mu}} \| \alpha_a J_a I_a F_a\rangle\right|^2$$

The magnetic dipole moment for hyperfine structure is

(Note: $|\hat{\vec{J}}| = \sqrt{J(J+1)}\,\hbar$, $\quad |\hat{\vec{I}}| = \sqrt{I(I+1)}\,\hbar$)

$$\hat{\vec{\mu}} = \hat{\vec{\mu}}_J + \hat{\vec{\mu}}_I = -\mu_B g_J\sqrt{J(J+1)} + \mu_B g_I'\sqrt{I(I+1)}$$

$$= -\frac{e}{2m_e}\left(g_J\hat{\vec{J}} - g_I'\hat{\vec{I}}\right) = -\mu_B\left(g_J\hat{\vec{J}} - g_I'\hat{\vec{I}}\right)$$

$$\therefore W_{sp}(F_a=1\to F_b=0) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{4\omega^3}{3\hbar c^5}\cdot\frac{\mu_B^2}{2F_a+1}\left|\langle\alpha_b J_b I_b F_b \| g_J\vec{J} - g_I'\vec{I} \| \alpha_a J_a I_a F_a\rangle\right|^2$$

$\langle\alpha_b J_b I_b F_b \| \hat{\vec{J}} \| \alpha_a J_a I_a F_a\rangle$

$$= (-1)^{J_b+I_b+F_a+1}\,\delta_{I_b I_a}\sqrt{(2F_a+1)(2F_b+1)}\begin{Bmatrix} F_b & 1 & F_a \\ J_a & I_a & J_b \end{Bmatrix}\langle J_b \| \hat{\vec{J}} \| J_a\rangle$$

$$= (-1)^{J_b+I_b+F_a+1}\,\delta_{I_b I_a}\sqrt{(2F_a+1)(2F_b+1)}\begin{Bmatrix} F_b & 1 & F_a \\ J_a & I_a & J_b \end{Bmatrix}\sqrt{(2J_a+1)(J_a+1)J_a}\;\delta_{J_a J_b}$$

$$= (-1)^{\frac{1}{2}+\frac{1}{2}+1+1}\times 1\times\sqrt{(2\times 1+1)(2\times 0+1)}\times\begin{Bmatrix} 0 & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{Bmatrix}\times\sqrt{(2\times\frac{1}{2}+1)(\frac{1}{2}+1)\frac{1}{2}}\times 1$$

$$= (-1)\times\sqrt{3}\times\frac{1}{\sqrt{6}}\times\sqrt{\frac{3}{2}} = -\frac{\sqrt{3}}{2}$$

$$\langle \alpha_b J_b I_b F_b \| \hat{\vec{I}} \| \alpha_a J_a I_a F_a \rangle$$

$$= (-1)^{J_b + I_a + F_b + 1} \delta_{J_b J_a} \sqrt{(2F_a+1)(2F_b+1)} \begin{Bmatrix} F_b & 1 & F_a \\ I_a & J_b & I_b \end{Bmatrix} \langle I_b \| \hat{\vec{I}} \| I_a \rangle$$

$$= (-1)^{J_b + I_a + F_b + 1} \delta_{J_b J_a} \sqrt{(2F_a+1)(2F_b+1)} \begin{Bmatrix} F_b & 1 & F_a \\ I_a & J_b & I_b \end{Bmatrix} \sqrt{(2I_a+1)(I_a+1)I_a}\, \delta_{I_b I_a}.$$

$$= (-1)^{\frac{1}{2}+\frac{1}{2}+0+1} \times 1 \times \sqrt{(2\times 1+1)(2\times 0+1)} \times \begin{Bmatrix} 0 & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{Bmatrix} \times \sqrt{(2\times\frac{1}{2}+1)(\frac{1}{2}+1)\frac{1}{2}} \times 1$$

$$= 1 \times 1 \times \sqrt{3} \times \frac{1}{\sqrt{6}} \times \sqrt{\frac{3}{2}} = \frac{\sqrt{3}}{2}$$

$$\therefore \langle \alpha_b J_b I_b F_b \| g_J \hat{\vec{J}} - g_I' \hat{\vec{I}} \| \alpha_a J_a I_a F_a \rangle$$

$$= -\frac{\sqrt{3}}{2} g_J - \frac{\sqrt{3}}{2} g_I' = -\frac{\sqrt{3}}{2}(g_J + g_I')$$

$$g_J = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)} = 1 + \frac{\frac{1}{2}\times\frac{3}{2} + \frac{1}{2}\times\frac{3}{2} - 0\times 1}{2\times\frac{1}{2}\times\frac{3}{2}} = 2$$

$$g_I' = g_I \cdot \frac{m_e}{M_p} = \frac{5.58}{1836} = 0.003$$

$$\therefore W_{sp}(F_a=1 \to F_b=0) = \frac{1}{4\pi\varepsilon_0} \cdot \frac{4\omega^3}{3\hbar c^5} \cdot \frac{\mu_B^2}{2F_a+1} \left[-\frac{\sqrt{3}}{2}(g_J + g_I)\right]^2$$

$$= \frac{(2\pi\nu)^3}{3\pi\varepsilon_0 \hbar c^5} \cdot \frac{e^2\hbar^2}{4m_e^2} \cdot \frac{1}{2F_a+1} \cdot \frac{3}{4} \cdot (g_J + g_I)^2$$

$$= \frac{\pi^2 \nu^3 e^2 \hbar}{2\varepsilon_0 c^5 m_e^2} \cdot \frac{1}{2F_a+1} \cdot (g_J + g_I)^2.$$

$$= \frac{3.14^2 \times (1420\times 10^6)^3 \times (1.6\times 10^{-19})^2 \times 6.626\times 10^{-34}/2\pi}{2\times 8.854\times 10^{-12} \times (3\times 10^8)^5 \times (9.1\times 10^{-31})^2} \cdot \frac{1}{2\times 1+1} \cdot (2+0.003)^2$$

$$= 2.85 \times 10^{-15}\ s^{-1}$$

Such small transition probability comes from two aspects:

① M1 transition

② Low frequency $\nu = 1420$ MHz.