(5) Quantization of the Field

The EM field is now quantized by the association of a quantum-mechanical harmonic oscillator with each mode $\vec{k}$ of the radiation field. The mode to which a quantum-mechanical operator refers is indicated by a subscript. $\hat{a}_k^+$ and $\hat{a}_k$ are the operators that creat and destroy a quantum of energy $\hbar\omega_k$ in the cavity EM-field mode of wavevector $\vec{k}$. These quanta are the photons of wavevector $\vec{k}$; the number of photons $\vec{k}$ excited in the cavity is determined by the eigenvalue n_k of the appropriate number operator $\hat{n}_k = \hat{a}^+\hat{a}$, and has possible values $0, 1, 2, \cdots$. The excitation level of a cavity mode $\vec{k}$ is determined by its eigenstate $|n_k\rangle$. The creation, destruction, and number operators for mode $\vec{k}$ applied to $|n_k\rangle$ obey the following rules:

$$\hat{a}_k |n_k\rangle = \sqrt{n_k}\, |n_k - 1\rangle \ , \quad \hat{a}|0\rangle = 0 . \tag{159}$$

$$\hat{a}_k^+ |n_k\rangle = \sqrt{n_k+1}\, |n_k + 1\rangle \tag{160}$$

$$\hat{n}_k |n_k\rangle = n_k\, |n_k\rangle \tag{161}$$

The states of the total radiation field in the cavity can be specified by the numbers of photons $n_{k_1}, n_{k_2}, n_{k_3}, \cdots$ excited in the complete set of cavity modes $\vec{k_1}, \vec{k_2}, \vec{k_3}, \cdots$.

In counting the normal modes, it must be remembered that for each wavevector $\vec{k}$, there are two independent directions of the mode polarization $\vec{e}_k$. Here we let the single symbol $\vec{k}$ stand for both the wavevector and polarization of a mode. With this convention, the phrase "a particular normal mode $\vec{k}$" implies that both the wavevector and polarization of the mode are specified.

A state of the total field can be written as $|n_{k1}, n_{k2}, n_{k3}, \cdots\rangle$ and is specified by a string of numbers. Since the different cavity modes are independent, the state of the total field can be written as a product of the states of the individual modes

$$|n_{k1}, n_{k2}, n_{k3}, \cdots\rangle = |n_{k1}\rangle |n_{k2}\rangle |n_{k3}\rangle \cdots \quad (162)$$

The states of the individual cavity modes will always be assumed normalized, and it follows that the total state of the field is also normalized.

An operator that refers to a particular normal mode $\vec{k}_i$ affects only the photons in that particular mode, for example,

$$\hat{a}^+_{k_i} |n_{k1}, n_{k2}, n_{k3}, \cdots, n_{ki}, \cdots\rangle = \sqrt{n_{ki}+1}\, |n_{k1}, n_{k2}, \cdots, n_{ki}+1, \cdots\rangle \quad (163)$$

Sometimes, we denote the states of the total field by the short-hand

$$|\{n_k\}\rangle = |n_{k1}\rangle |n_{k2}\rangle |n_{k3}\rangle \cdots \quad (164)$$

The symbols $\{n_k\}$ here denote the complete set of numbers that specify the excitation levels of all the harmonic oscillators associated with the cavity modes. There is always an infinite numbers of oscillators. The $|\{n_k\}\rangle$ form a complete set of states for the EM field in the cavity when every member n_{ki} of the set $\{n_k\}$ is allowed to take on the values zero and all the positive integers.

The classical vector potential $\vec{A}_k$ and $\vec{A}_k^*$ for cavity mode $\vec{k}$ expressed in term of P_k and Q_k by Eqs. (122) and (123) are converted into quantum-mechanical operators expressed in terms of $\hat{P}_k$ and $\hat{q}_k$:

$$\vec{A}_k = (4\varepsilon_0 V\omega_k^2)^{-1/2}(\omega_k Q_k + iP_k)\vec{e}_k \longrightarrow$$
$$(4\varepsilon_0 V\omega_k^2)^{-1/2}(\omega_k \hat{q}_k + i\hat{P}_k)\hat{\vec{e}}_k = (\hbar/2\varepsilon_0 V\omega_k)^{\frac{1}{2}}\, \hat{a}_k \hat{\vec{e}}_k \quad (165)$$

$$\vec{A}_k^* = (4\varepsilon_0 V\omega_k^2)^{-1/2}(\omega_k Q_k - iP_k)\vec{e}_k \longrightarrow$$
$$(4\varepsilon_0 V\omega_k^2)^{-1/2}(\omega_k \hat{q}_k - i\hat{p}_k)\hat{\vec{e}}_k = (\hbar/2\varepsilon_0 V\omega_k)^{\frac{1}{2}}\, \hat{a}_k^+ \hat{\vec{e}}_k \quad (166)$$

The relationships given by Eqs. (127) and (128) have been used above.

The transition from classical to quantum mechanics thus consists of the replacement of the classical Fourier coefficients $\vec{A}_k$ and $\vec{A}_k^*$ by the destruction operator $\hat{a}_k$ and the creation operator $\hat{a}_k^+$, multiplied by a numerical factor and a unit vector. The quantum-mechanical expression for the total vector potential is obtained by substitution of Eqs. (165) and (166) into Eq. (117):

$$\hat{\vec{A}} = \sum_k (\hbar/2\varepsilon_0 V\omega_k)^{1/2}\hat{\vec{e}}_k\{\hat{a}_k \exp(-i\omega_k t + i\vec{k}\cdot\hat{\vec{r}}) + \hat{a}_k^+ \exp(i\omega_k t - i\vec{k}\cdot\hat{\vec{r}})\} \quad (167)$$

The corresponding results for the electric and magnetic field operators $\hat{\vec{E}}_k$ and $\hat{\vec{B}}_k$ associated with mode $\vec{k}$ are obtained by making similar substitutions in Eqs. (119) and (120):

$$\hat{\vec{E}}_k = i(\hbar\omega_k/2\varepsilon_0 V)^{1/2}\hat{\vec{e}}_k\{\hat{a}_k\exp(-i\omega_k t + i\vec{k}\cdot\hat{\vec{r}}) - \hat{a}_k^+\exp(i\omega_k t - i\vec{k}\cdot\hat{\vec{r}})\} \quad (168)$$

$$\hat{\vec{B}}_k = i(\hbar/2\varepsilon_0 V\omega_k)^{1/2}\vec{k}\times\hat{\vec{e}}_k\{\hat{a}_k\exp(-i\omega_k t + i\vec{k}\cdot\hat{\vec{r}}) - \hat{a}_k^+\exp(i\omega_k t - i\vec{k}\cdot\hat{\vec{r}})\}. \quad (169)$$

The operators for the total transverse electric and magnetic fields are

$$\hat{\vec{E}}_T = \sum_{\vec{k}} \hat{\vec{E}}_k, \quad \hat{\vec{B}} = \sum_k \hat{\vec{B}}_k \quad (170)$$

The field energy becomes $\bar{\mathcal{E}}_k = \frac{1}{2}\int_{cavity} \overline{\langle n_k|\varepsilon_0 \hat{\vec{E}}_k\cdot\hat{\vec{E}}_k + \frac{1}{\mu_0}\hat{\vec{B}}_k\cdot\hat{\vec{B}}_k|n_k\rangle}\, dV.$

It can be proven that $\bar{\mathcal{E}}_k = (n_k + \frac{1}{2})\hbar\omega_k$. (171)

The Hamitonian operator for the total EM field in the cavity is

$$\hat{H}_f = \sum_{\vec{k}} \hbar\omega_k(\hat{a}_k^+\hat{a}_k + \tfrac{1}{2}) \quad (172)$$

The total energy of the radiation for cavity state $|\{n_k\}\rangle$ is

$$\bar{\mathcal{E}} = \sum_{\vec{k}} \bar{\mathcal{E}}_k = \sum_{\vec{k}} (n_k + \tfrac{1}{2})\hbar\omega_k. \quad (173)$$

2. Radiative Transition Probabilities

Radiative transitions are to deal with the interaction between an atom and a radiation field. Armed with the quantized radiation field, here let us treat the radiative transitions with full quantum mechanics, i.e., a quantized atom interacts with a quantized radiation field.

When the atom and the radiation field have interaction, the total Hamiltonian operator is given by

$$\hat{H} = \hat{H}_a + \hat{H}_f + \hat{H}_{int} \qquad (174)$$

where $\hat{H}_a$, $\hat{H}_f$, and $\hat{H}_{int}$ are the Hamiltonian operators of the atom, the radiation field, and the interaction. The idea behind the total Hamiltonian operator $\hat{H}$ is that the atom and the radiation field are treated as a whole system, and both are described by quantum mechanics.

$$\hat{H}_a = \sum_i \frac{\hat{\vec{P}}_i^{\,2}}{2\mu} + U \qquad (175)$$

$$\hat{H}_f = \sum_k \hbar \omega_k \left(\hat{a}_k^+ \hat{a}_k + \frac{1}{2}\right) \qquad (176)$$

where $\sum_i$ is to take sum for all electrons inside the atom, U includes all interactions that determine the atomic state, like the Coulomb interaction between the nucleus and electrons, electrostatic interaction between electrons, spin-orbit coupling, nuclear spin-electron coupling, and constant external field influence, etc. $\hat{a}_k^+$ is the creation operator, $\hat{a}_k$ is the annihilation operator (also called destruction operator).

Assume $\hat{H}_a$ has eigenstates $|a\rangle$ and $|b\rangle$:

$$\hat{H}_a|a\rangle = E_a|a\rangle, \quad \hat{H}_a|b\rangle = E_b|b\rangle. \tag{177}$$

$\hat{H}_f$ has eigenstates $|n_{k\lambda}\rangle$ and $|n_{k\lambda}\pm 1\rangle$

$$\hat{H}_f|n_{k\lambda}\rangle = (n_{k\lambda}+\tfrac{1}{2})\hbar\omega_{k\lambda}|n_{k\lambda}\rangle \tag{178}$$
$$\hat{H}_f|n_{k\lambda}\pm 1\rangle = (n_{k\lambda}\pm 1+\tfrac{1}{2})\hbar\omega_{k\lambda}|n_{k\lambda}\rangle$$

Note that both $\hat{H}_a$ and $\hat{H}_f$ are independent of time.

Let
$$\hat{H}_0 = \hat{H}_a + \hat{H}_f \tag{179}$$

$$|A\rangle = |a\rangle|n_{k\lambda}\rangle = |a, n_{k\lambda}\rangle \tag{180}$$
$$|B\rangle = |b\rangle|n_{k\lambda}\pm 1\rangle = |b, n_{k\lambda}\pm 1\rangle \tag{181}$$

Note: The reason that we can write $|A\rangle$ and $|B\rangle$ as Eqs.(180-181) is that $\hat{H}_a$ and $\hat{H}_f$ are independent of each other.

Thus, we have

$$\hat{H}_0|A\rangle = [E_a + (n_{k\lambda}+\tfrac{1}{2})\hbar\omega_{k\lambda}]|A\rangle = E_A|A\rangle \tag{182}$$
$$\hat{H}_0|B\rangle = [E_b + (n_{k\lambda}\pm 1+\tfrac{1}{2})\hbar\omega_{k\lambda}]|B\rangle = E_B|B\rangle \tag{183}$$

where
$$E_A \equiv E_a + (n_{k\lambda}+\tfrac{1}{2})\hbar\omega_{k\lambda} \tag{184}$$
$$E_B \equiv E_b + (n_{k\lambda}\pm 1+\tfrac{1}{2})\hbar\omega_{k\lambda} \tag{185}$$

From Eqs.(182) and (183), we know the eigenstates

$$|\psi_A(t)\rangle = |A\rangle e^{-iE_A t/\hbar}, \quad |\psi_B(t)\rangle = |B\rangle e^{-iE_B t/\hbar} \tag{186}$$

satisfy the Schrödinger equation:

$$i\hbar\frac{d}{dt}|\psi_N(t)\rangle = \hat{H}_0|\psi_N(t)\rangle, \quad N=A, B \tag{187}$$

The total Hamiltonian operator $\hat{H}$ is given by

$$\hat{H} = \hat{H}_0 + \hat{H}_{int} \tag{188}$$

The idea is to treat $\hat{H}_{int}$ as a time-dependent perturbation to $\hat{H}_0$, and then use perturbation theory (1st order) to derive the transition probability. Similar to what we did in the semi-classical theory, the initial conditions are

$$C_A(0) = 1, \quad C_{N \neq A}(0) = 0, \quad \text{i.e., the system is at } |\psi_A(t=0)\rangle \tag{189}$$

For Schrödinger equation

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle, \tag{190}$$

the state

$$|\psi(t)\rangle = \sum_N C_n(t)|\psi_N(t)\rangle \tag{191}$$

The probability of the system in state $|\psi_B(t)\rangle$ at time t is given by $|C_B(t)|^2$

Similar to the derivation we did in semi-classical theory,

$$i\hbar \frac{d}{dt} C_B^{(1)}(t) = C_A^{(0)}(t)\, e^{i(E_B - E_A)t/\hbar} \langle B|\hat{H}_{int}|A\rangle$$

$$\Rightarrow i\hbar \frac{d}{dt} C_B^{(1)}(t) = e^{i(E_B - E_A)t/\hbar} \langle B|\hat{H}_{int}|A\rangle. \tag{192}$$

$$\therefore C_B^{(1)}(t) = \frac{1}{i\hbar}\int_0^t e^{i(E_B - E_A)t/\hbar} \langle B|\hat{H}_{int}|A\rangle\, dt \tag{193}$$

Therefore, the transition probability from $|A\rangle$ to $|B\rangle$ is

$$P_{A\to B}(t) = |C_B(t)|^2 = |C_B^{(1)}(t)|^2 = \frac{1}{\hbar^2}\left|\int_0^t e^{i(E_B-E_A)t/\hbar} \langle B|\hat{H}_{int}|A\rangle dt\right|^2 \tag{194}$$

The transition probability per unit time:

$$W_{A\to B}(t) = \frac{d P_{A\to B}(t)}{dt} = \frac{1}{\hbar^2}\frac{d}{dt}\left|\int_0^t e^{i(E_B-E_A)t/\hbar} \langle B|\hat{H}_{int}|A\rangle\right|^2 \tag{195}$$

To derive the transition probability, we must write out the interaction Hamiltonian operator $\hat{H}_{int}$. For the entire system of the atom interacting with the radiation field, another way to write the total Hamiltonian $\hat{H}$ is

$$\hat{H} = \hat{H}_f + \sum_i \left\{ \frac{[\hat{\vec{P}}_i - e\hat{\vec{A}}(t)]^2}{2\mu} - \frac{e}{\mu}\hat{\vec{S}}_i \cdot \hat{\vec{B}} \right\} + U \qquad (196)$$

where $\hat{\vec{S}}_i \cdot \hat{\vec{B}}$ represents the interaction of the spin magnetic moment of the electron with the oscillating magnetic field of the plane wave.

Eq. (196) can be further derived to

$$\hat{H} = \hat{H}_f + \underbrace{\left(\sum_i \frac{\hat{\vec{P}}_i^{\,2}}{2\mu} + U\right)}_{\hat{H}_a} - \sum_i \left(\frac{e}{\mu}\hat{\vec{P}}_i \cdot \hat{\vec{A}} + \frac{e}{\mu}\hat{\vec{S}}_i \cdot \hat{\vec{B}} - \frac{e^2}{2\mu}(\hat{\vec{A}}(t))^2\right) \qquad (197)$$

$$\therefore \hat{H}_{int} = -\sum_i \left[\frac{e}{\mu}\hat{\vec{P}}_i \cdot \hat{\vec{A}}(t) + \frac{e}{\mu}\hat{\vec{S}}_i \cdot \hat{\vec{B}} - \frac{e^2}{2\mu}(\hat{\vec{A}}(t))^2\right] \qquad (198)$$

The first two terms in Eq. (198) depend on $\hat{\vec{A}}$ linearly, and the third one depends on it quadratically. With normal light sources, the intensity is sufficiently low that the effect of the $\hat{\vec{A}}^2$ term can be neglected.

To compare the 1st and the 2nd terms, $\hat{\vec{S}}$ is on the order of $\hbar$, $\hat{\vec{B}}$ is of the order of kA. Thus, $(k = 2\pi/\lambda)$ (p is momentum)

$$\frac{\hat{\vec{S}}_i \cdot \hat{\vec{B}}}{\hat{\vec{P}}_i \cdot \hat{\vec{A}}} \approx \frac{\hbar k A_0}{p A_0} = \frac{\hbar k}{p}$$

$\hbar/p$ is, at most, of the order of atomic dimension (Bohr radius a_0 $= 0.5\,\text{Å}$)

$k = 2\pi/\lambda$, $\therefore$

$$\frac{\hat{\vec{S}}_i \cdot \hat{\vec{B}}}{\hat{\vec{P}}_i \cdot \hat{\vec{A}}} \simeq \frac{a_0}{\lambda} \ll 1 \qquad (199)$$

The large inequality comes from the fact light wavelength is ~100nm or more for most cases we study.

Thus, $\frac{e^2}{2\mu}\hat{\vec{A}}^2(t) \ll \frac{e}{\mu}\hat{\vec{S}}_i\cdot\vec{B} \ll \frac{e}{\mu}\hat{\vec{P}}_i\cdot\vec{B}$ (200)

Let us only consider the $\hat{\vec{P}}_i\cdot\hat{\vec{A}}(t)$ term for now.

$$\hat{H}_{int} = -\sum_i\left[\frac{e}{\mu}\hat{\vec{P}}_i\cdot\hat{\vec{A}}(t)\right] \quad (201)$$

Substitute Eq. (167) into Eq. (201),

$$\hat{H}_{int} = -\sum_i\sum_K \frac{e}{\mu}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_K}}\left(\vec{e}_K\cdot\hat{\vec{P}}_i\right)\left\{\hat{a}_K e^{-i\omega_K t+i\vec{k}\cdot\hat{\vec{r}}} + \hat{a}_K^+ e^{i\omega_K t-i\vec{k}\cdot\hat{\vec{r}}}\right\} \quad (202)$$

Some complication occurs here. Since we define the atom and the radiation field as an entire system, the interactions between them are regarded as internal interaction. Because no external interaction is introduced, the entire system energy (the atom + the radiation field) should be conservative. This means that the total Hamiltonian $\hat{H}$ is independent of time in the representation $\{\Psi_N(t)=|N\rangle e^{-iE_N t/\hbar},\ N=A, B, \cdots\}$

Since $\hat{H}_a$ and $\hat{H}_f$ are independent of time, $\hat{H}_{int}$ should be turned into time-independent operator, i.e., set $t=0$:

$$\hat{H}_{int} = -\frac{e}{\mu}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_K}}\left(\vec{e}_K\cdot\hat{\vec{P}}\right)\left\{\hat{a}_K e^{i\vec{k}\cdot\hat{\vec{r}}} + \hat{a}_K^+ e^{-i\vec{k}\cdot\hat{\vec{r}}}\right\} \quad (203)$$

Note: Here we only consider single ~~electron~~ atom and single mode radiation field to simplify the $\hat{H}_{int}$ to Eq. (203).

This is called Schrödinger representation (or picture), i.e., operators are independent of time, but all time-dependence is included in the state vector.

Appendix: Schrödinger, Heisenberg, and Interaction Representation

These are three different ways to handle the time factor.

(1) Schrödinger Representation (Picture)

All observable operators are independent of time.

The time factors are included only in the state vectors.

For example, when we use $\{\psi_N = |N\rangle e^{-iE_N t/\hbar},\ N=A,B,\cdots\}$

as the Representation, we are working in the Schrödinger Representation.

This is because $\hat{H} = \hat{H}_a + \hat{H}_f + \hat{H}_{int}$ all operators are time-independent, and $e^{-iE_N t/\hbar} = e^{-i(\hat{H}_a+\hat{H}_f)t/\hbar}$ includes all time dependence factors.

(2) Heisenberg Representation (Picture)

All state vectors are independent of time,

Time factors are included only in the operators.

Heisenberg representation is opposite to Schrödinger representation.

(3) Interaction Representation (Picture)

Schrödinger and Heisenberg Representations are two extremes,

While the Interaction Representation is in the middle —

Part of time factors are included in the state vector,

and other part of time factors are included in the operators.

For example, in the semi-classical theory, we use

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = \left(\hat{H}_0 + \hat{H}'(t)\right)|\psi(t)\rangle$$

where $\hat{H}_0$ is time-independent, but both $\hat{H}'(t)$ and $|\psi(t)\rangle$ are time-dependent. All representations are equivalent, i.e., giving the same results, but they can make problems simplified.

Since $\hat{H}_{int}$ is time-independent, we can obtain from the 1st order perturbation equation Eq. (193) the coefficient $C_B^{(1)}(t)$:

$$C_B^{(1)}(t) = \frac{1}{i\hbar} \cdot \frac{1}{i(E_B-E_A)/\hbar} \langle B|\hat{H}_{int}|A\rangle e^{i(E_B-E_A)t/\hbar}\Big|_0^t$$

$$= \frac{-1}{\hbar} \frac{e^{i(E_B-E_A)t/\hbar}-1}{(E_B-E_A)/\hbar} \langle B|\hat{H}_{int}|A\rangle \qquad (204)$$

Since $$\frac{e^{i(E_B-E_A)t/\hbar}-1}{(E_B-E_A)/\hbar} = e^{i(E_B-E_A)t/2\hbar} \frac{\sin[(E_B-E_A)t/2\hbar]}{(E_B-E_A)/2\hbar} \qquad (205)$$

$\therefore$ The transition probability from $|A\rangle \to |B\rangle$ is (Eq. (194))

$$P_{A\to B} = |C_B(t)|^2 = |C_B^{(1)}(t)|^2 = \frac{|\langle B|\hat{H}_{int}|A\rangle|^2}{\hbar^2} \frac{\sin^2[(E_B-E_A)t/2\hbar]}{[(E_B-E_A)/2\hbar]^2} \qquad (206)$$

Since $$\lim_{t\to\infty} \frac{\sin^2[(E_B-E_A)t/2\hbar]}{[(E_B-E_A)/2\hbar]^2} = \pi t\, \delta[(E_B-E_A)/2\hbar]$$

$$\boxed{\delta(ax) = \frac{1}{a}\delta(x)} \qquad = 2\pi\hbar t\, \delta(E_B-E_A) \qquad (207)$$

Thus, for large t,

$$P_{A\to B} = \frac{2\pi t}{\hbar} |\langle B|\hat{H}_{int}|A\rangle|^2 \delta(E_B-E_A) \qquad (208)$$

From Eqs. (184) and (185), we have

$$E_B - E_A = [E_b + (n_k \pm 1 + \tfrac{1}{2})\hbar\omega_k] - [E_a + (n_k + \tfrac{1}{2})\hbar\omega_k]$$

$$= E_b - E_a \pm \hbar\omega_k$$

$$\therefore P_{A\to B} = \frac{2\pi t}{\hbar} |\langle B|\hat{H}_{int}|A\rangle|^2 \delta(E_b - E_a \pm \hbar\omega_k) \qquad (209)$$

The transition probability per unit time is

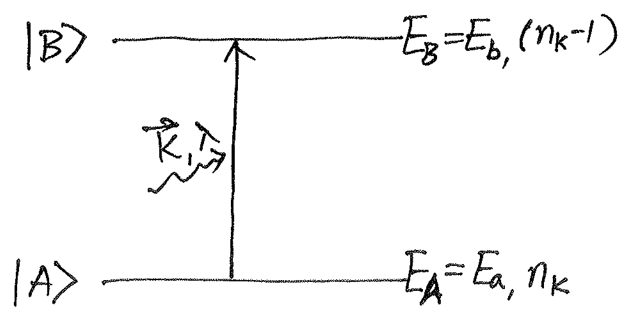
$$W_{A\to B} = \frac{dP_{A\to B}}{dt} = \frac{2\pi}{\hbar}|\langle B|\hat{H}_{int}|A\rangle|^2\delta(E_b - E_a \pm \hbar\omega_k) \quad (210)$$

Again, as we stated in the semi-classical theory, Eqs. (209) and Eq. (210) are valid only for conditions of

$$\frac{\hbar}{|E_b - E_a|} \ll t \ll \frac{\hbar}{|\langle B|\hat{H}_{int}|A\rangle|} \quad (211)$$

Now let us analyze the transition in two different processes: absorption and emission.

(1) For absorption, the initial state is the lower state $|a\rangle$ with n_k photons; the final state is the upper state $|b\rangle$ with $(n_k - 1)$ photons. From Eq. (203)



$$\langle B|\hat{H}_{int}|A\rangle = \langle b, n_k - 1|\hat{H}^{(-)}_{int}|a, n_k\rangle$$

$$= -\frac{e}{\mu}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_k}}\langle n_k - 1|\hat{a}_k|n_k\rangle \cdot \langle b|\hat{\vec{e}}_k \cdot \hat{\vec{p}}\, e^{i\vec{k}\cdot\hat{\vec{r}}}|a\rangle$$

$$= -\frac{e}{\mu}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_k}}\sqrt{n_k}\,\langle b|\hat{\vec{e}}_k \cdot \hat{\vec{p}}\, e^{i\vec{k}\cdot\hat{\vec{r}}}|a\rangle \quad (212)$$

Substitute Eq. (212) into Eq. (210),

$$W_{A\to B} = \frac{2\pi}{\hbar}\frac{e^2}{\mu^2}\frac{\hbar}{2\varepsilon_0 V\omega_k}n_k|\langle b|\hat{\vec{e}}_k \cdot \hat{\vec{p}}\, e^{i\vec{k}\cdot\hat{\vec{r}}}|a\rangle|^2\delta(E_b - E_a - \hbar\omega_k)$$

$$= \frac{\pi e^2 n_k}{\mu^2\varepsilon_0 V\omega_k}|\langle b|\hat{\vec{e}}_k \cdot \hat{\vec{p}}\, e^{i\vec{k}\cdot\hat{\vec{r}}}|a\rangle|^2\delta(E_b - E_a - \hbar\omega_k) \quad (213)$$

This is absorption probability per unit time, i.e., absorption rate for atom with single mode radiation field.

(2) For emission, initial state $|a, n_K\rangle$, final state $|b, n_K+1\rangle$.

Similarly, $\langle B|\hat{H}_{int}|A\rangle = \langle b, n_K+1|\hat{H}^{(+)}_{int}|a, n_K\rangle$

$$= -\frac{e}{\mu}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_K}}\langle n_K+1|\hat{a}_K^{+}|n_K\rangle\langle b|\hat{\vec{e}}_K\cdot\hat{\vec{P}}e^{-i\vec{K}\cdot\hat{\vec{r}}}|a\rangle$$

$$= -\frac{e}{\mu}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_K}}\sqrt{n_K+1}\,\langle b|\hat{\vec{e}}_K\cdot\hat{\vec{P}}e^{-i\vec{K}\cdot\hat{\vec{r}}}|a\rangle \quad (214)$$

$$\therefore W_{A\to B} = \frac{2\pi}{\hbar}\frac{e^2}{\mu^2}\cdot\frac{\hbar}{2\varepsilon_0 V\omega_K}(n_K+1)|\langle b|\hat{\vec{e}}_K\cdot\hat{\vec{P}}e^{-i\vec{K}\cdot\hat{\vec{r}}}|a\rangle|^2\delta(E_b-E_a+\hbar\omega_k)$$

$$= \frac{\pi e^2(n_K+1)}{\mu^2\varepsilon_0 V\omega_K}|\langle b|\hat{\vec{e}}_K\cdot\hat{\vec{P}}e^{-i\vec{K}\cdot\hat{\vec{r}}}|a\rangle|^2\delta(E_b-E_a+\hbar\omega_k) \quad (215)$$

This is the emission probability per unit time., i.e., the emission rate for atom interacting with single mode radiation field.

Note: Eq.(215) has a term that is independent of photon number n_K. This is to say even when there is no radiation field, there is still emission — spontaneous emission.

$$W^{sp}_{A\to B} = \frac{\pi e^2}{\mu^2\varepsilon_0 V\omega_K}|\langle b|\hat{\vec{e}}_K\cdot\hat{\vec{P}}e^{-i\vec{K}\cdot\hat{\vec{r}}}|a\rangle|^2\delta(E_b-E_a+\hbar\omega_k) \quad (216)$$

$$W^{st}_{A\to B} = \frac{\pi e^2 n_K}{\mu^2\varepsilon_0 V\omega_K}|\langle b|\hat{\vec{e}}_K\cdot\hat{\vec{P}}e^{\mp i\vec{K}\cdot\hat{\vec{r}}}|a\rangle|^2\delta(E_b-E_a\pm\hbar\omega_k) \quad (217)$$

sp — spontaneous, st — stimulated.