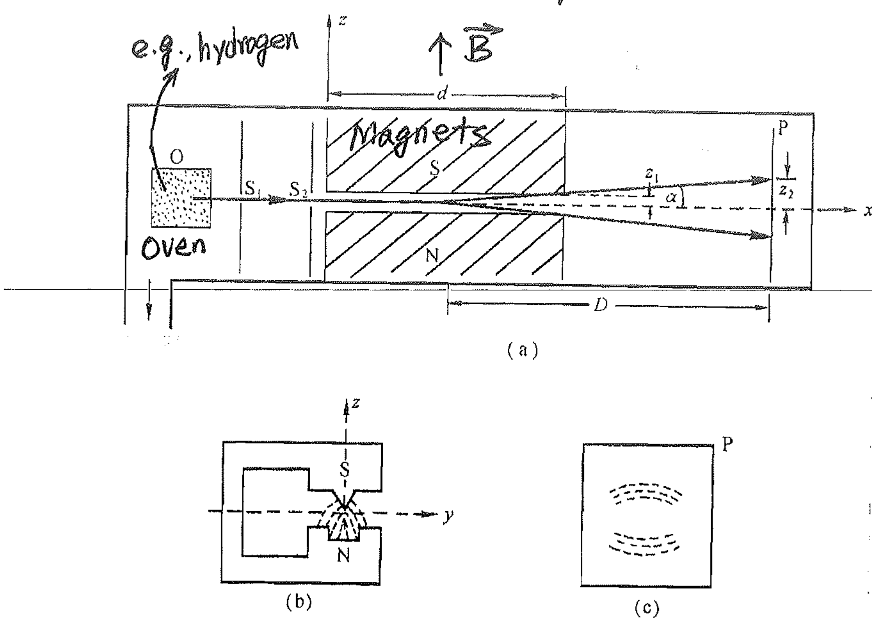
(4) Stern – Gerlach Experiment. (1921)



(a) (b) (c)

Hydrogen atoms in container O is heated to become vapor. In thermal dynamic equilibrium, the atom velocity v is given by $\frac{1}{2}mv^2 = \frac{3}{2}k_BT$, where k_B is Boltzmann Constant, and T is the temperature.

Hydrogen atoms coming out of container O pass through slits S_1 and S_2, thus, we select the atoms going along x-direction. The magnetic field provided by two magnets are

$$\frac{\partial B_z}{\partial x} = \frac{\partial B_z}{\partial y} = 0, \quad \text{but } \frac{\partial B_z}{\partial z} \neq 0.$$

Thus, in inhomogeneous magnetic field, a magnetic moment will experience force: $F_z = \mu_z \frac{\partial B_z}{\partial z}$.

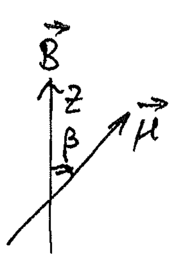
For atoms entering the magnetic field with x-direction velocity, they experience force in z-direction (perpendicular), thus,

$$\begin{cases} x = vt \\ z_1 = \frac{1}{2}at^2 = \frac{1}{2}\frac{F_z}{m}t^2 \end{cases}$$

$\Rightarrow$ tilted angle $\alpha = \arctan\left(\frac{dz}{dx}\right)_d = \arctan\left(\frac{F_z t}{mv}\right)_d = \arctan\left(\frac{F_z d}{mv^2}\right)$

After leaving magnets, atoms will go along straight line with α to the screen.

$$\Rightarrow z_2 = \mu_z \cdot \frac{\partial B_z}{\partial z} \cdot \frac{d \cdot D}{mv^2} = \mu_z \cdot \frac{\partial B_z}{\partial z} \cdot \frac{d \cdot D}{3k_BT}.$$



Since $\mu_z = \mu \cos\beta$, if $\cos\beta$ can be arbitrary, then μ_z is not quantized, and z_2 won't be quantized.

Only when $\cos\beta$ is quantized, i.e., the spatial direction of μ is quantized, then μ_z is quantized $\Rightarrow Z_z$ is quantized.
Therefore, the experimental results will show whether μ_z is quantized. Indeed, Z_z is quantized — only two directions.
$\Rightarrow$ Conclusion: angular momentum / magnetic moment is quantized in magnetic field.

* However, until that point, the only two-direction cannot be explained by theory. Because for certain l, the spatial direction m_l has $2l+1$ directions. Since l is an integer, $2l+1$ has to be an odd number, not even!!!

$\Rightarrow$ This led to the electron spin hypothesis!

(5) Electron spin and magnetic moment.

* To explain the Stern-Gerlach experiement (even number of splitting), Uhlenbeck and Goudsmit proposed a hypothesis in 1925: an electron is not a point charge, but owns spin motion besides the orbital angular momentum.

* The intrinsic spin angular momentum ($\vec{S}$) $\vec{S}$ (of an electron) is given by

$$|\vec{S}| = \sqrt{s(s+1)}\,\hbar, \quad s = \frac{1}{2}$$

Along z-direction, the component of spin angular momentum is

$$S_z = m_s \hbar, \quad m_s = \pm\frac{1}{2}.$$

* The spin magnetic moment corresponding to the spin angular momentum is given by

$$\mu_s = -\sqrt{s(s+1)}\,\mu_B\, g_s$$

$$\mu_{s,z} = -m_s\, \mu_B\, g_s \qquad \boxed{g_s = 2}$$

This g_s can be derived from Dirac's relativity QM: $g_s \overset{\text{exact}}{\doteq} 2$

* Lande g-factor: for an arbitrary angular momentum j, the corresponding magnetic moment μ_j is given by

$$\begin{cases} \mu_j = -\sqrt{j(j+1)}\, g_j\, \mu_B \\ \mu_{j,z} = -m_j\, g_j\, \mu_B \end{cases}.$$

$$\therefore\ g_j = \frac{\text{Measured } \mu_z \text{ in } \mu_B \text{ unit}}{\text{angular momentum projection } l_z \text{ in } \hbar \text{ unit}}$$

(6) Spin-orbit coupling:

* To explain the Stern-Gerlach experiment, and the fine structure observed in hydrogen atomic spectra, we must an important interaction inside the atom, besides the major Coulomb force betwee the nucleus and the electron. This new force is the magnetic interaction between electron's orbit and electron spin, which is called the electron spin-orbit coupling.

* This coupling can be understood in the following three ways:

①. The movement of electron around the nucleus can also be regarded as the nucleus moving around the electron (in the electron-rest coordinates). Therefore, it produces current

$$i_Z = Zef = \frac{Zev}{2\pi r}$$

(v - speed, e - electron charge, Z - nuclear number, r - orbit radius)

Therefore, the magnetic field produced by the current at the electron is

$$B = \frac{1}{4\pi\varepsilon_0}\frac{2\pi i}{c^2 r} = \frac{1}{4\pi\varepsilon_0}\frac{Zev}{c^2 r^2}$$

Written in vector: $\vec{B} = \frac{1}{4\pi\varepsilon_0}\frac{Ze}{c^2 r^3}(-\vec{v})\times\vec{r} = \frac{1}{4\pi\varepsilon_0}\frac{Ze}{E_0 r^3}\vec{l}$

where $\vec{l} = m_e\vec{r}\times\vec{v}$, $E_0 = m_e c^2$ is the rest energy of electron.

Electron has spin magnetic moment $\vec{\mu}_s$, so it will interact with the magnetic field and have potential energy

$$U = -\vec{\mu}_s \cdot \vec{B}$$

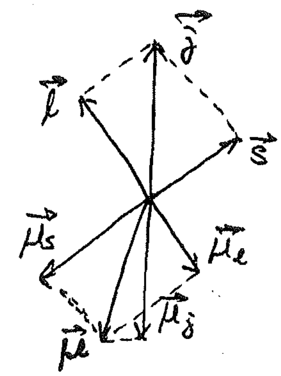
Substituting $\mu_s = -\sqrt{s(s+1)}\, g_s \mu_B$, $|\vec{s}| = \sqrt{s(s+1)}\,\hbar$, and $\vec{B}$ into eq., we obtain: $U = \frac{1}{4\pi\varepsilon_0}\frac{Z g_s \mu_B e}{E_0 \hbar r^3}\vec{s}\cdot\vec{l}$ — This is the equation in the coordinates with the electron at rest.

When turn it into the coordinates with the nucleus at rest, there is a correction factor $\frac{1}{2}$. Therefore,

$$U = \frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{Z g_s \mu_B e}{m_e c^2 \hbar r^3}\vec{s}\cdot\vec{l}, \quad \boxed{g_s = 2} \quad \boxed{\mu_B = \frac{e\hbar}{2m_e}}$$

② Another view is to say the spin-orbit coupling is the magnetic interaction between the orbital magnetic moment $\vec{\mu}_l$ and the spin magnetic moment $\vec{\mu}_s$ of the electron — analogy to the interaction between two small magnet bars.

③. More accurate explanation is from QM — the coupling of two angular momentums and cause the additional energy corrections:



$$\vec{j} = \vec{s} + \vec{l}$$

$$\therefore \quad \vec{j}^2 = \vec{s}^2 + \vec{l}^2 + 2\vec{s}\cdot\vec{l}$$

Eigenvalues of $\hat{\vec{j}}^2$ are $j(j+1)\hbar^2$,

where $j = |l+s|, |l+s|-1, \cdots, |l-s|$.

Eigenvalues of j_z are $m_j\hbar$, where $m_j = j, j-1, \cdots, -j$.

and $m_j = m_l + m_s$.

The interaction of the electron spin-orbit coupling can be expressed as an operator:

$$\Delta\hat{H}_{ls} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{Z g_s \mu_B e}{2 m_e c^2 \hbar r^3}\hat{s}\cdot\hat{l}.$$

Thus, the total Hamiltonian operator: $\hat{H} = \hat{H}_0 + \Delta\hat{H}_{ls}$

where $\hat{H}_0 = \frac{\hat{p}^2}{2\mu} + \left(-\frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r}\right)$.

Using time-dependent perturbation theory (an approach), we can derive the energy shift ΔE_{ls} caused by this spin-orbit coupling interaction by deriving the mean value of $\Delta \hat{H}_{ls}$ in the state ψ_{nls} (i.e., small perturbation doesn't cause the change ~~of wave function ψ_{nls}):~~

$$\Delta E_{ls} = \langle \Delta \hat{H}_{ls} \rangle = \int \psi_{nls}^* \, \Delta \hat{H}_{ls} \, \psi_{nls} \, d^3r .$$

The result is:

$$\begin{cases} \Delta E_{ls} = \dfrac{(\alpha Z)^4 m_e c^2}{4 n^3} \dfrac{[j(j+1) - l(l+1) - s(s+1)]}{l(l+1/2)(l+1)}, & l \neq 0 \\ \Delta E_{ls} = 0, & \text{when } l = 0 \end{cases}$$

Where $\alpha = \dfrac{e^2}{4\pi\varepsilon_0 \hbar c} \approx \dfrac{1}{137}$ is the fine structure constant (dimensionless) important!

$$\overline{\hat{\vec{S}} \cdot \hat{\vec{l}}} = \frac{1}{2} \overline{(\hat{j}^2 - \hat{s}^2 - \hat{l}^2)} = \frac{1}{2} [j(j+1) - s(s+1) - l(l+1)] \hbar^2$$

(7) As mentioned above, after the coupling of two angular momentums, the magnetic moment corresponding to the total angular momentum is given by $\mu_j = -\sqrt{j(j+1)}\, g_j \mu_B$, $\mu_{j,z} = -m_j g_j \mu_B$.

QM shows that g_j (the Lande g-factor) is given by

$$g_j = \frac{3}{2} + \frac{1}{2}\left(\frac{\hat{s}^2 - \hat{l}^2}{\hat{j}^2}\right) = \frac{3}{2} + \frac{1}{2}\left[\frac{s(s+1) - l(l+1)}{j(j+1)}\right].$$

This also stands true if an atom has many electrons, and electrons for an total spin S, total orbital L, and L-S Coupling is true, then $(\vec{J} = \vec{L} + \vec{S}) \rightarrow$ $g_J = \dfrac{3}{2} + \dfrac{1}{2}\left[\dfrac{S(S+1) - L(L+1)}{J(J+1)}\right].$

(8) Rules for angular momentum coupling:

$$\vec{J} = \vec{L} + \vec{S}$$

Quantum number: $J = |L+S|, |L+S|-1, \cdots, |L-S|$

$$m_J = J, J-1, \cdots, -J$$

Different J will corresponding to different energy correction, so the originally degenerate energy levels may become non-degenerate, or partially non-degenerate.

* To express different energy states, we use the following terms: $^{2S+1}L_J$

spin quantum number

orbital angular momentum quantum number

total angular momentum quantum number.

Capitalized letters S, L, J are used to denote the overall electrons' effects. If it is a single electron, like in hydrogen-like atom or ions, we may use small letters: s, l, j.

For example, for H-atom ground state

$^2S_{1/2}$,

$s = \frac{1}{2} \longrightarrow 2S+1 = 2 \Rightarrow S = 1/2$

$l = 0 \longrightarrow S \to l = 0$

$j = \frac{1}{2}$.

$l = 0, 1, 2, 3, 4 \ldots$

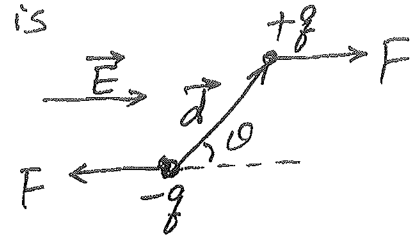
$\Rightarrow$ S, P, D, F, G...

s, p, d, f, g, ...

Appendix Dipole moment:

(1) Electric Dipole Moment: definition is

$$\vec{D} = q\vec{d}$$



In homogeneous electric field,

the electric dipole will experience a moment of force (torque)

$$\vec{\tau} = \vec{d} \times \vec{F} = \vec{d} \times (q\vec{E}) = (q\vec{d}) \times \vec{E}$$

$$\Rightarrow \quad \vec{\tau} = \vec{D} \times \vec{E}$$

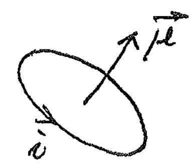
The potential energy of the electric dipole moment in the homogeneous electric field is given by the work done by this moment of force: $U_E = \int_{\pi/2}^{\theta} \tau d\theta = -DE\cos\theta$.

Here we define the potential energy $=0$ for $\theta = 90°$.

Thus, the potential energy $U_E = -\vec{D} \cdot \vec{E}$

(2) Magnetic dipole moment: defined as

$$\vec{\mu} = iS\vec{n}_0 = i\vec{S}.$$



In homogeneous magnetic field,

the magnetic dipole will experience a moment of force (torque),

$$\vec{\tau} = \vec{\mu} \times \vec{B}$$

Similar to the electric dipole moment case, the potential energy of the magnetic moment in the homogeneous magnetic field is given by $U_B = -\vec{\mu} \cdot \vec{B}$

Any force can be written as the gradiant of a potential energy:

$$\vec{F} = -\vec{\nabla} U = -\left(\frac{\partial U}{\partial x}\vec{i} + \frac{\partial U}{\partial y}\vec{j} + \frac{\partial U}{\partial z}\vec{k}\right).$$

∴ Z component of force acting on a magnetic moment is

$$F_z = -\frac{\partial U}{\partial z} = \mu_x \frac{\partial B_x}{\partial z} + \mu_y \frac{\partial B_y}{\partial z} + \mu_z \frac{\partial B_z}{\partial z}.$$

(3) Force, moment of Force, momentum, and angular momentum

① Force causes the change of momentum, i.e.,

$$\vec{F} = \frac{d}{dt}(m\vec{v}) = \frac{d\vec{p}}{dt}.$$

In another format:

$$\vec{F}\cdot\Delta t = \Delta(m\vec{v}) = \Delta\vec{p} = \vec{p}_{Final} - \vec{p}_{initial}$$

i.e., impulse (force x time) is ~~x~~ equal to the change amount of momentum.

② Moment of force causes the change of angular momentum,

i.e., $$\vec{\tau} = \vec{r}\times\frac{d}{dt}(m\vec{v}) = \frac{d\vec{l}}{dt}, \qquad \vec{l} = \vec{r}\times\vec{p}$$

$\because \vec{l} = \vec{r}\times\vec{p}$,

$$\therefore \frac{d\vec{l}}{dt} = \frac{d(\vec{r})}{dt}\times m\vec{v} + \vec{r}\times\frac{d(m\vec{v})}{dt} = \vec{r}\times\frac{d(m\vec{v})}{dt} = \vec{r}\times\vec{F} = \vec{\tau}.$$

$\because \frac{d\vec{r}}{dt} = \vec{v}$, $\therefore \frac{d\vec{r}}{dt}\times m\vec{v} = 0$

In other format: $\vec{\tau}\ast\Delta t = \Delta\vec{l} = \vec{l}_{Final} - \vec{l}_{initial}$

i.e., Moment of impulse (i.e., $\vec{\tau}\ast\Delta t = \vec{r}\times(\vec{F}\Delta t)$) is equal to the change amount of angular momentum.

(9) Explanation of Stern-Garlach Experiment

* In the Stern-Gerlach experiment, the hydrogen atoms are in the ground state: $n=1$, $l=0$. For the single atom, it has a spin angular momentum $s=1/2$.

The electron spin-orbit coupling: $\vec{j} = \vec{s} + \vec{l} \Rightarrow j = l+s = 1/2$
$= |l-s| = 1/2$

i.e., for the H ground state, there is only one total angular momentum $j = 1/2$.

* According to the rules we described in last lecture, the ground state term is $^{2S+1}L_J \Rightarrow {}^{2\times\frac{1}{2}+1}S_{1/2} \Rightarrow {}^2S_{1/2}$

If we add n at the left side of this term, then the H ground state is: $1\,{}^2S_{1/2}$ (format: $\underline{\underline{n\,{}^{2S+1}L_J}}$)

* For the coupled total angular moment, the atom has an electron magnetic moment $\mu_j = -\sqrt{j(j+1)}\, g_j\, \mu_B$

and its z-component $\mu_{j,z} = m_j\, g_j\, \mu_B$

For H: $1\,{}^2S_{1/2}$: $s=1/2$, $l=0$, $j=1/2$, $n=1$, $m_j = \pm 1/2$

$$\therefore g_j = \frac{3}{2} + \frac{1}{2}\left[\frac{\frac{1}{2}\times(\frac{1}{2}+1) - 0(0+1)}{\frac{1}{2}\times(\frac{1}{2}+1)}\right] = \frac{3}{2} + \frac{1}{2} = 2.$$

$$\therefore \mu_j = -\frac{\sqrt{3}}{2}\times 2\mu_B = -\sqrt{3}\,\mu_B$$

$$\mu_{j,z} = \pm\frac{1}{2}\times 2\mu_B = \pm\mu_B.$$

i.e., μ_z has two distinct values: $+\mu_B$ and $-\mu_B$.

* In S-G experiment, $Z_2 = \mu_z \frac{\partial B_z}{\partial z}\frac{d \cdot D}{mv^2} = \pm\mu_B \frac{\partial B_z}{\partial z}\cdot\frac{d\cdot D}{3k_BT}$

$\therefore$ Hydrogen beam splits to two beams under inhomogeneous magnetic field: one goes up and another goes down.

* Some concrete experimental data in S-G experiment:

$\frac{\partial B_z}{\partial z} = 10\ T/m$, $d = 1m$, $D = 2m$, $T = 400k$, $k_B = 1.38 \times 10^{-23}\ J/k$

$\mu_B = 0.9274 \times 10^{-23}\ J/T$. We get:

$$Z_2 = \pm \mu_B \frac{\partial B_z}{\partial z} \cdot \frac{d \cdot D}{3 k_B T}$$

$$= \pm 0.9274 \times 10^{-23} \times 10 \times \frac{1 \times 2}{3 \times 1.38 \times 10^{-23} \times 400}$$

$$\simeq \pm 1.12 \times 10^{-2} m = \pm 1.12\, cm.$$

* Tablized S-G experimental results for different atoms

Atoms	Ground State	g	$m \cdot g$	Pattern
Zn, Cd, Hg, Pd	$^1S_0$	—	0	|
Sn, Pb	$^3P_0$	—	0	|
H, Li, Na, K, Cu, Ag, Au	$^2S_{1/2}$	2	± 1	| |
Tl	$^2P_{1/2}$	2/3	$\pm 1/3$	| |
	$^3P_2$	3/2	$\pm 3, \pm \frac{3}{2}, 0$	| | | | |
O	$^3P_1$	3/2	$\pm \frac{3}{2}, 0$	| | |
	$^3P_0$	—	0	|

* Stern-Gerlach experiment was a milestone experiment in atomic/Quantum physics, as it proved three things:

(1) Angular momentum is spatially quantized.

(2) an electron has spin angular momentum, and $S = 1/2$

(3) The electron spin magnetic moment $\mu_{s,z} = \pm \mu_B$, and $g_s = 2$.