Fundamentals of Spectroscopy for (Optical) Remote Sensing
Homework #2 (Quantum Mechanics)

1. In lidar remote sensing application, sometimes our return photon signal is so weak that you can practically consider the photons coming back one by one. However, when a Fabry-Perot etalon is used in the lidar receiver, it actually works as a narrowband daytime filter to pass the wanted frequency while rejecting lots of unwanted frequencies. Could you explain this single-photon Fabry-Perot etalon phenomenon from Quantum Mechanics point of view? (You may check optics book or on the web about how a Fabry-Perot etalon works.)

2. Please summarize the postulates and principles of Quantum Mechanics in your own words.

3. In a one-dimensional problem, consider a particle whose wave function is

$$\psi(x) = N\frac{e^{ip_0 x/\hbar}}{\sqrt{x^2 + a^2}}$$

where a and p_0 are real constants and N is a normalization coefficient.
(1) Determine N so that $\psi(x)$ is normalized.
(2) The position of the particle is measured. What is the probability of finding a result between $-\frac{a}{\sqrt{3}}$ and $+\frac{a}{\sqrt{3}}$?
(3) Calculate the mean value of the momentum of a particle that has $\psi(x)$ for its wave function.

4. The energy operator for harmonic oscillator is

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}\kappa\hat{x}^2 = \frac{1}{2m}\left(\hat{p}^2 + m^2\omega_0^2\hat{x}^2\right),$$

where $\omega_0 = \sqrt{\kappa/m}$ is the intrinsic angular frequency of the harmonic oscillator. The normalized wave function for the ground state of the oscillator is given by

$$\psi_0(\xi) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4} e^{-\xi^2/2},$$

where $\xi = \left(\sqrt{m\omega_0/\hbar}\right)x$. (1) Please compute the mean values of the oscillator's momentum and energy. (2) Is this wave function an eigen function of the energy operator $\hat{H}$ (also called Hamilton operator)? If yes, how much is the eigen value?

5. For a particle moving within a range of x =0 to x = a, its normalized wave function is given by

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right), & 0 < x < a \\ 0, & elsewhere \end{cases}$$

Is this wave function an eigen function of momentum? Is it an eigen function of kinetic energy? If so, how much is the corresponding eigen value?

6. A particle's momentum and the position of itself have the commutation relation $[\hat{x}, \hat{p}_x] = i\hbar$. However, the variables between different particles are commuted. In other words, the following relationship exists for particles 1 and 2:

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}, \qquad (i, j = 1,2).$$

Try to prove that the operators $\hat{x}_1 - \hat{x}_2$ and $\hat{p}_1 + \hat{p}_2$ are commuted with each other.

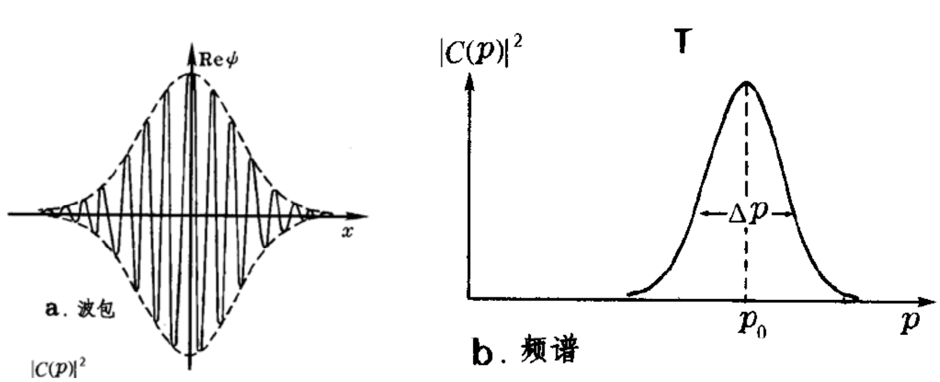
7. (* Extra problem - Not required to finish but you are encouraged to give it a try!)
Please use the following conditions and steps to derive Heisenberg uncertainty relation.

(1) Assume a particle described by a wave packet (see Figues) has a Gaussian wave function $\psi(x) = Ae^{-ax^2/2}e^{+ip_0x/\hbar}$, where A is the normalization factor $A = (\pi/a)^{-1/4}$. The uncertainty of the coordinate x should be given by $\Delta x = \sqrt{\overline{x^2}}$. Try to derive its value from the formula $\overline{x^2} = \int_{-\infty}^{+\infty} \Psi^*(x) x^2 \Psi(x) dx$.

(2) When expressed in the momentum representation, the wave function is given by $C(p) = \int_{-\infty}^{+\infty} \Psi(x) e^{-ipx/\hbar} \frac{dx}{\sqrt{2\pi\hbar}} = \sqrt{\frac{1}{a\hbar}} A e^{-(p-p_0)^2/2a\hbar^2}$. The uncertainty of the momentum p should be given by $\Delta p = \sqrt{\overline{(p-p_0)^2}}$. Try to derive its value using the formula $\overline{(p-p_0)^2} = \int_{-\infty}^{+\infty} C^*(p)(p-p_0)^2 C(p) dp$.

(3) After obtaining Δx and Δp, take the product of Δx and Δp to derive the Heisenberg uncertainty relation.

(4) Try to derive $\Delta p = \sqrt{\overline{(p-p_0)^2}}$ in the x representation using the formula $\overline{(p-p_0)^2} = \int_{-\infty}^{+\infty} \Psi^*(x)(\hat{p}-p_0)^2 \Psi(x) dx$, where $\hat{p} = -i\hbar\frac{d}{dx}$. Do you get the same result as step (3)? If yes, why?



a. 波包 b. 频谱