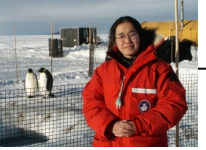
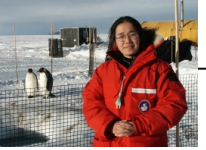


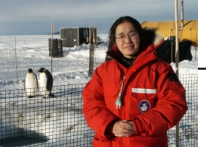
Lecture 23. Lidar Error and Sensitivity Analysis (2)

- Derivation of Errors
- Background vs. Noise
- Sensitivity Analysis
- Summary

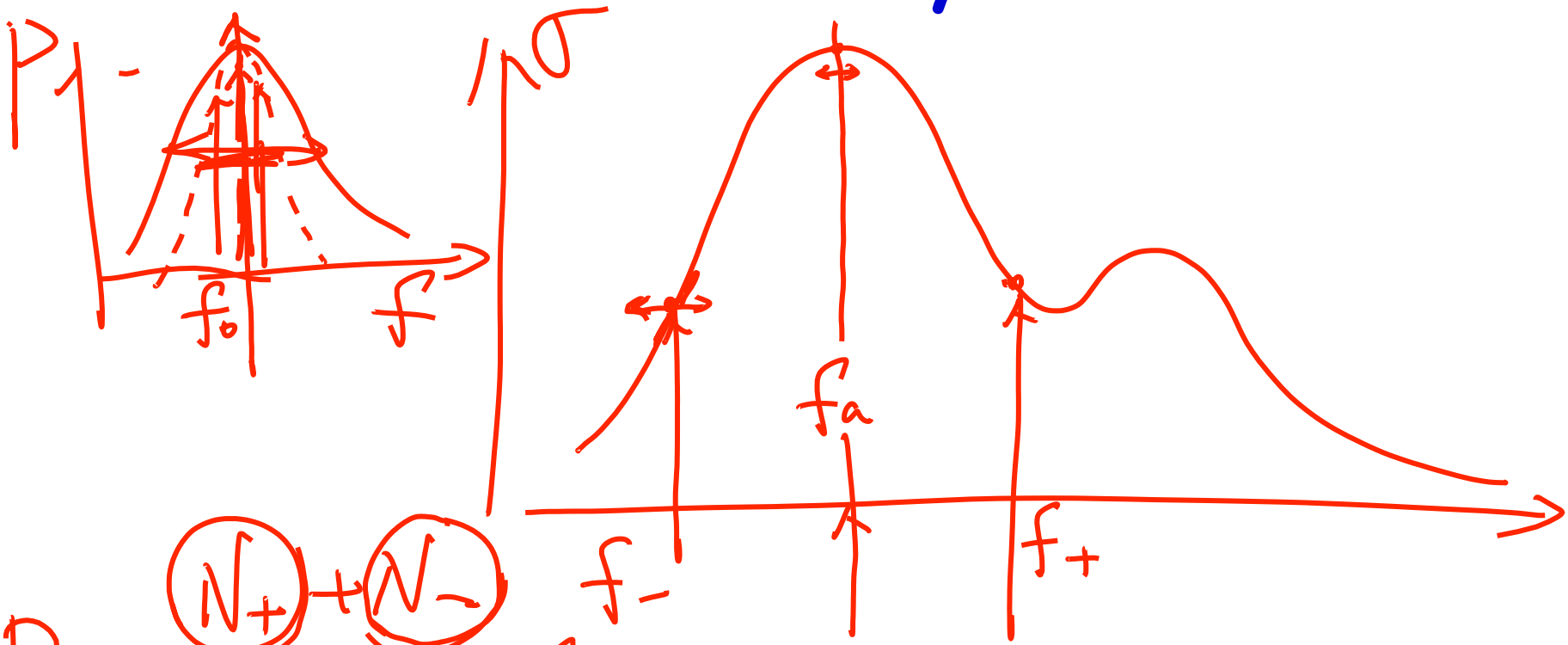


Accuracy vs. Precision in Lidar Measurements

- The precision errors caused by the random error sources like laser frequency jitter, linewidth fluctuation, and electronic jitter can be improved by integrating more shots together - sacrifice of temporal resolution, but may not be improved by integrating bins.
- Random error sources could lead to both random and systematic measurement errors. For example, laser central frequency jitter in the 3-freq ratio technique can lead to warm temperature bias (systematic error) in addition to random errors.
- The differentiation of metric ratio method described in later slides can apply to both systematic and random errors, depending on the nature of the errors. Error sources could be systematic bias or random jitter, and measurement errors could also be systematic or random errors.
- For example, the chirp in f_a is a systematic error source if it is not counted, while the jitter in f_a is a random error. → See manual drawing



Laser Central Frequency Jitter: Random Error vs. Systematic Bias

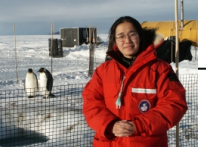


$$R_T = \frac{N_+ + N_-}{N_a}$$

$\rightarrow T \nearrow$

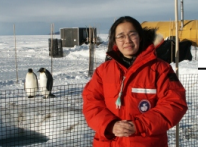
$f_L = f_a,$

$f_\pm = f_L \pm \Delta f$



Propagation of Errors

- Propagation of Errors is an important aspect in lidar error analysis. This is because the temperature, wind, backscatter coefficient, etc. that we want to determine are dependent variables that are a function of one or more different measured variables (e.g., photon counts, laser frequency and linewidth). We must know how to propagate or carry over the uncertainties in the measured variables to determine the uncertainty in the dependent variables.
- For example, photon noise causes the uncertainty in the measured photon counts, then the photon count uncertainty leads to the uncertainty in the temperature and wind ratios R_T and R_W, which will result in errors in the inferred temperature T and wind W. -- Error propagation procedure
- Basic rules for propagation of error can be found in many textbooks, e.g., addition, subtraction, multiplication, division, product of power, and mixture of them, along with many other complicated functions.
- We will introduce a universal procedure through the use of differentials of the corresponding ratios R_T and R_W as illustrated below. This method is mathematically based on the Taylor expansion.



Error Analysis Procedure

- We use the temperature error derivation for 3-freq Na lidar as an example to explain the error analysis procedure using a differentiation method.

$R_T(T, f_a, f_\pm, \sigma_L, V_R \cdots) \rightarrow T(R_T, f_a, f_\pm, \sigma_L, V_R \cdots)$

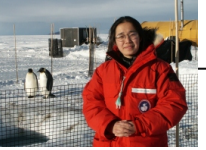
- For 3-frequency technique, we have the temperature ratio

$$R_T = \frac{\sigma_{eff}(f_+) + \sigma_{eff}(f_-)}{\sigma_{eff}(f_a)} = \frac{N(f_+) + N(f_-)}{N(f_a)}$$

- Through this ratio R_T or further through the effective cross-section, the temperature T is an implicit function of R_T, laser frequencies f_a, f_+, f_-, laser linewidth σ_L, radial wind, etc. Each parameter could have some uncertainty or error, leading to errors in the measured temperature.

- Therefore, the temperature error is given by the derivatives

$$\Delta T = \frac{\partial T}{\partial R_T}\Delta R_T + \frac{\partial T}{\partial f_a}\Delta f_a + \frac{\partial T}{\partial f_\pm}\Delta f_\pm + \frac{\partial T}{\partial \sigma_L}\Delta\sigma_L + \frac{\partial T}{\partial v_R}\Delta v_R + \textit{higher-order terms}$$



Differentiation Method

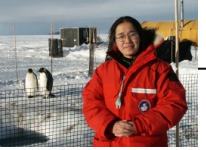
- The root-mean-square (rms) temperature error is given by

$$(\Delta T)_{rms} = \sqrt{\left(\frac{\partial T}{\partial R_T}\Delta R_T + \frac{\partial T}{\partial f_a}\Delta f_a + \frac{\partial T}{\partial f_\pm}\Delta f_\pm + \frac{\partial T}{\partial \sigma_L}\Delta \sigma_L + \frac{\partial T}{\partial v_R}\Delta v_R\right)^2}$$

- If the error sources are independent from each other, then the means of cross terms are zero. Then we have

$$(\Delta T)_{rms} = \sqrt{\left(\frac{\partial T}{\partial R_T}\Delta R_T\right)^2 + \left(\frac{\partial T}{\partial f_a}\Delta f_a\right)^2 + \left(\frac{\partial T}{\partial f_\pm}\Delta f_\pm\right)^2 + \left(\frac{\partial T}{\partial \sigma_L}\Delta \sigma_L\right)^2 + \left(\frac{\partial T}{\partial v_R}\Delta v_R\right)^2}$$

- The above error equation indicates that many laser parameters and radial wind errors could affect the inferred temperature because they all influence the effective cross sections. In the meantime, photon noise can cause uncertainty in the ratio R_T, resulting in temperature error.



Error Derivation: Implicit Differentiation

- How to derive the error coefficients, like $\frac{\partial T}{\partial R_T}, \frac{\partial T}{\partial f_a}, etc.$?
- We may use the implicit differentiation through R_T as below:

$$\boxed{\begin{aligned}\Delta T = \Delta R_T\left(\frac{\partial R_T}{\partial R_T}\Big/\frac{\partial R_T}{\partial T}\right) + \Delta f_a\left(\frac{\partial R_T}{\partial f_a}\Big/\frac{\partial R_T}{\partial T}\right) + \Delta f_\pm\left(\frac{\partial R_T}{\partial f_\pm}\Big/\frac{\partial R_T}{\partial T}\right)\\ + \Delta\sigma_L\left(\frac{\partial R_T}{\partial \sigma_L}\Big/\frac{\partial R_T}{\partial T}\right) + \Delta v_R\left(\frac{\partial R_T}{\partial v_R}\Big/\frac{\partial R_T}{\partial T}\right)\end{aligned}}$$

- For the photon-noise induced temperature error,

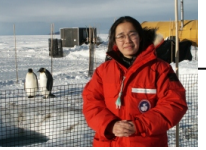
$$\Delta T = \frac{1}{\partial R_T / \partial T}\cdot \Delta R_T = \frac{R_T}{\partial R_T / \partial T}\cdot\frac{\Delta R_T}{R_T} = \frac{1}{S_T}\cdot\frac{\Delta R_T}{R_T}$$

$\propto \frac{1}{SNR}$

$\propto \frac{1}{S_T \cdot SNR}$

where S_T is called the sensitivity of R_T to temperature $S_T = (\partial R_T / \partial T)/R_T$

- The relative error of R_T can be derived in terms of measured signal and background photon counts (see later slides).



Derivation of Error Coefficients

- The temperature error coefficient can be derived numerically

$$\frac{R_T}{\partial R_T / \partial T} = \frac{R_T}{[R_T(T+\delta T) - R_T(T)]/\delta T}$$

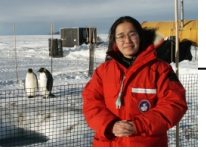
- Two approaches to derive the above numerical solution:

(1) One way is to use the equation of R_T in terms of cross sections. You don't have to go through the entire simulation process each time when you change the temperature, but just calculate the R_T from the effective cross section.

$$R_T = \frac{\sigma_{eff}(f_+,T) + \sigma_{eff}(f_-,T)}{\sigma_{eff}(f_a,T)}$$

(2) Another way is to use the equation of R_T in terms of photon counts, and then go through the entire simulation procedure to re-compute R_T for each new temperature. This method is more universal than the first approach, because not all cases could have a R_T written in terms of pure physical cross sections.

$$R_T = \frac{N(f_+,T) + N(f_-,T)}{N(f_a,T)}$$



Derivation of $\Delta R_T/R_T$

- We use 2-freq ratio technique of Na lidar as an example to derive the relative error $\Delta R_T / R_T$.

2-freq temperature ratio is defined as

$$R_T = \frac{N_{f_c}}{N_{f_a}} \quad (1)$$

$N_{f_c} = N_T - \bar{B}$

Using differentiation method, we have

$$\Delta R_T = \frac{\partial R_T}{\partial N_{f_c}} \Delta N_{f_c} + \frac{\partial R_T}{\partial N_{f_a}} \Delta N_{f_a} = \frac{1}{N_{f_a}} \Delta N_{f_c} - \frac{N_{f_c}}{N_{f_a}^{\ 2}} \Delta N_{f_a} \quad (2)$$

$\Delta N_{f_c} \neq \sqrt{N_{f_c}}$

Combining Eq. (1) with Eq. (2), we have

$$\frac{\Delta R_T}{R_T} = \frac{\Delta N_{f_c}}{N_{f_c}} - \frac{\Delta N_{f_a}}{N_{f_a}} \quad (3)$$

$= \sqrt{N_T}$

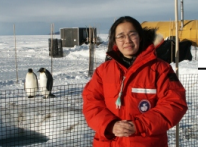
Regarding the errors from two frequencies are uncorrelated, we have

$$\left(\frac{\Delta R_T}{R_T}\right)_{rms} = \sqrt{\overline{\left(\frac{\Delta N_{f_c}}{N_{f_c}} - \frac{\Delta N_{f_a}}{N_{f_a}}\right)^2}} = \sqrt{\overline{\left(\frac{\Delta N_{f_c}}{N_{f_c}}\right)^2} + \overline{\left(\frac{\Delta N_{f_a}}{N_{f_a}}\right)^2}} \quad (4)$$

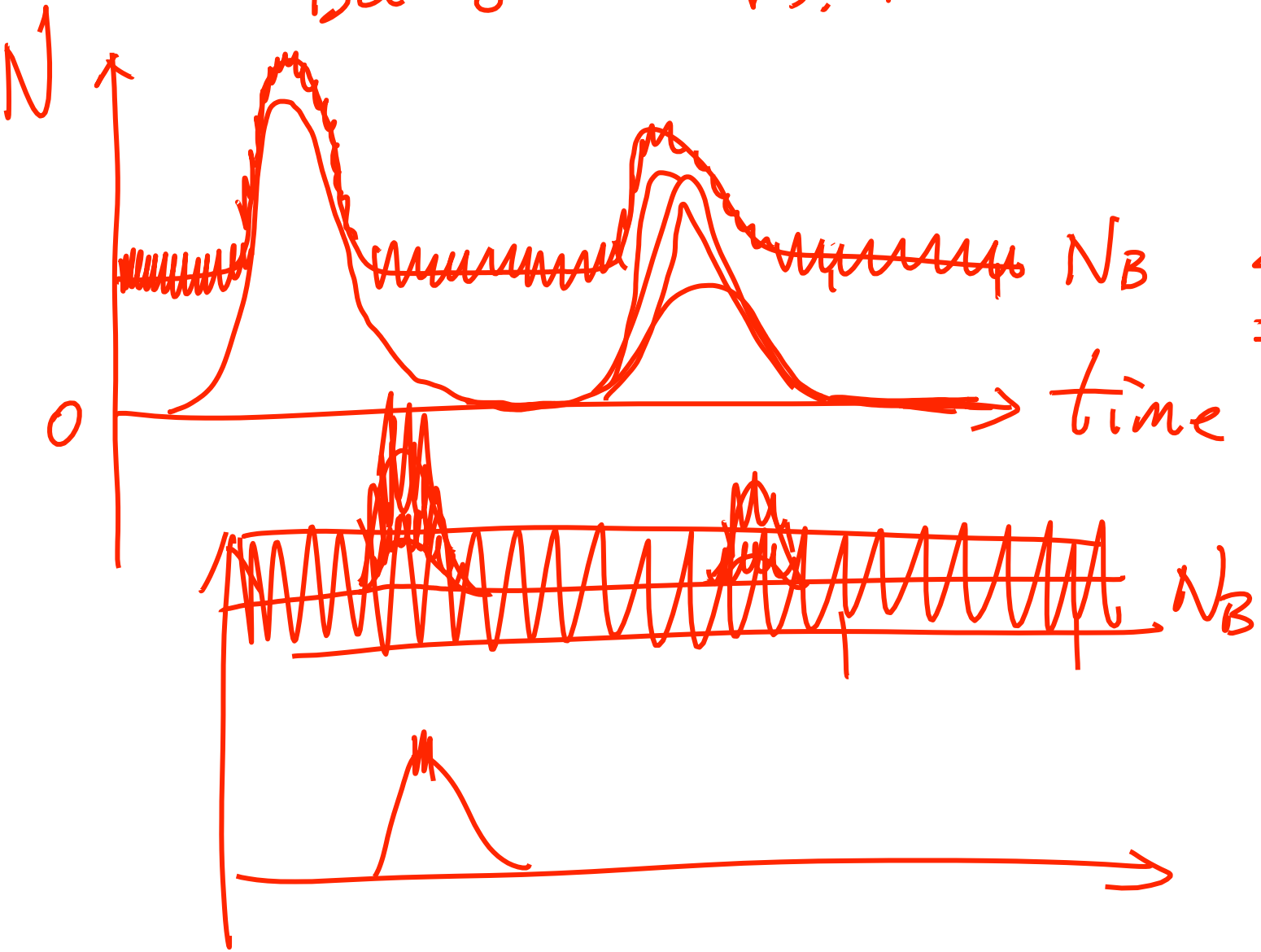
$= \sqrt{N_{f_c} + B}$

Considering the signal photon counts are derived by subtracting the background counts from the total photon counts, the photon count uncertainty is given by

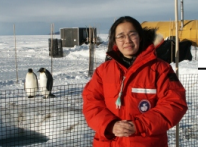
$$\overline{\left(\Delta N_{f_c}\right)^2} = N_{f_c} + B, \ \overline{\left(\Delta N_{f_a}\right)^2} = N_{f_a} + B \quad (5)$$



Background vs. noise



$$\Delta N_B = \sqrt{N_B}$$



Derivation of $\Delta R_T/R_T$ Cont'd

- Substituting Eq. (5) into Eq. (4) and considering Eq. (1), we obtain

$$\left(\frac{\Delta R_T}{R_T}\right)_{rms} = \sqrt{\frac{N_{f_c}+B}{N_{f_c}^{\ 2}} + \frac{N_{f_a}+B}{N_{f_a}^{\ 2}}} = \sqrt{\frac{R_T N_{f_a}+B}{\left(R_T N_{f_a}\right)^2} + \frac{N_{f_a}+B}{N_{f_a}^{\ 2}}} \qquad (6)$$

- Some algebra derivation leads us to the final result

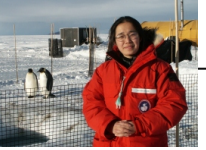
$$\boxed{\left(\frac{\Delta R_T}{R_T}\right)_{rms} = \frac{\left(1+\frac{1}{R_T}\right)^{1/2}}{\left(N_{f_a}\right)^{1/2}}\left[1+\frac{B}{N_{f_a}}\frac{\left(1+\frac{1}{R_T^{\ 2}}\right)}{\left(1+\frac{1}{R_T}\right)}\right]^{1/2}} \qquad (7)$$

- If we change the expression to SNR of the peak frequency channel, then we have an approximate expression as below:

$$\left(\frac{\Delta R_T}{R_T}\right)_{rms} \approx \frac{1}{SNR_{f_a}}\sqrt{1+\frac{1}{R_T}} \qquad (8)$$

where SNR is defined as

$$SNR_{f_a} \equiv \frac{N_{f_a}}{\Delta N_{f_a}} = \frac{N_{f_a}}{\sqrt{N_{f_a}+B}} \qquad (9)$$



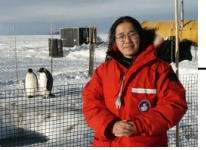
Temperature Error Due to Photon Noise

- Integrating above equations together, we obtain the equation for the temperature error caused by photon noise as below:

$$\Delta T = \frac{R_T}{\partial R_T / \partial T} \cdot \frac{\Delta R_T}{R_T}$$

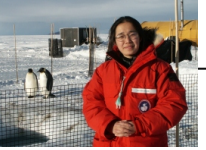
$$= \frac{R_T}{[R_T(T+\delta T) - R_T(T)]/\delta T} \cdot \frac{\left(1+\frac{1}{R_T}\right)^{1/2}}{\left(N_{f_a}\right)^{1/2}} \left[1 + \frac{B}{N_{f_a}} \frac{\left(1+\frac{2}{R_T{}^2}\right)}{\left(1+\frac{1}{R_T}\right)}\right]^{1/2}$$

- The photon counts in the above equation can be written in terms of signal to noise ratio (SNR), if it is more convenient or desirable for some analyses.



Sensitivity Analysis

- Sensitivity Analysis is part of a complete lidar simulation and error analysis. It is to answer the question how sensitive the measurement errors depend on lidar, atomic, and atmospheric parameters.
- We will show how several key lidar parameters affect measurement errors: (1) laser rms linewidth, and (2) laser central frequency.
- These factors are closely related to instrument design, while other factors like cross-talk between temperature and wind error, Hanle effect, etc. can be addressed independent of instrument design.
- Sensitivity Analysis helps define the requirements on instruments, e.g., linewidth and its stability, central frequency offset and stability, frequency shift and its stability.
- One of the main purposes for instrument design is to ensure that the accuracy or precision errors caused by lidar parameter uncertainties are less than the desired measurement errors, like 1 m/s and 1 K for wind and temperature, and also less than the errors caused by photon noise.



Methodology

(1) Start with the ratio metrics, like R_T or R_W, that are expressed through effective cross-section, e.g., for 3-frequency technique, as

$$R_T = \frac{\sigma_{eff}(f_+) + \sigma_{eff}(f_-)}{\sigma_{eff}(f_a)} \qquad R_W = \frac{\sigma_{eff}(f_-)}{\sigma_{eff}(f_+)}$$

Thus, R_T and R_W are functions of temperature, wind, laser linewidth, laser central frequency, AOM frequency shift, and atomic parameters, etc.

$$R_T(T, V_R, \sigma_L, f_L, f_{AOM}, \ldots), R_W(T, V_R, \sigma_L, f_L, f_{AOM}, \ldots)$$

(2) As an example, the temperature error caused by the uncertainty in laser RMS width should be derived as

Taylor expansion →

$$\Delta T = \frac{\partial T}{\partial \sigma_{rms}} \cdot \Delta\sigma_{rms} = \frac{\partial R_T / \partial \sigma_{rms}}{\partial R_T / \partial T} \cdot \Delta\sigma_{rms} + (?)\frac{\partial^2 T}{\partial \sigma_{rms}^2}(\Delta\sigma_{rms})^2$$

Based on principle of derivative of implicit function:

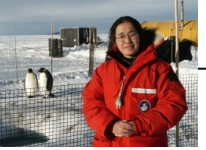
-- T is an implicit function of σ_{rms} through R_T.

The temperature error coefficient is derived as

$$\frac{\partial T}{\partial \sigma_{rms}} = \frac{\partial R_T / \partial \sigma_{rms}}{\partial R_T / \partial T} \propto \frac{1}{S_T}$$

$$\frac{\partial T}{\partial \sigma_{rms}} = \frac{[R_T(\sigma_{rms} + \delta\sigma_{rms}) - R_T(\sigma_{rms})]/\delta\sigma_{rms}}{[R_T(T + \delta T) - R_T(T)]/\delta T}$$

(K/MHz)



Methodology Cont'd

(3) Considering the nonlinear dependence of error coefficient on laser linewidth, actual temperature error can be calculated as (for larger uncertainty)

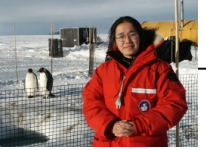
$$\Delta T = \frac{[R_T(\sigma_{rms} + \Delta\sigma_{rms}) - R_T(\sigma_{rms})]/\Delta\sigma_{rms}}{[R_T(T + \delta T) - R_T(T)]/\delta T} \cdot \Delta\sigma_{rms} \qquad \text{(K)}$$

(4) Both temperature error and error coefficient can be computed for each operating point, e.g., T = 200 K, V_R = 0 m/s, σ_{rms} = 60 MHz, etc. The operating points may be varied, e.g., try σ_{rms} of 10, 20, 30, 40, 60, 100 MHz, or T = 150, 200, 250 K, or V_R = -20, 0, +20 m/s.

(5) Such method can be applied to the wind metric R_W.

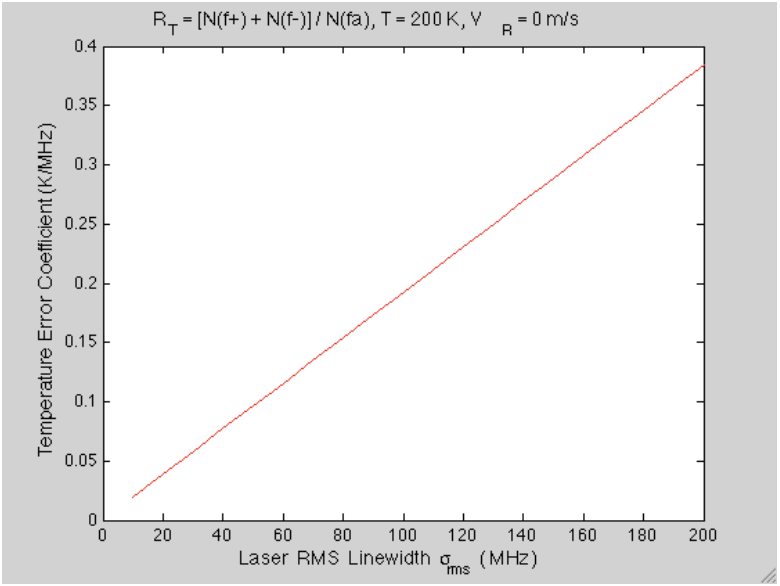
(6) Also, similar method can be used on laser central frequency, AOM frequency shift, etc. for both temperature error and wind error analyses.

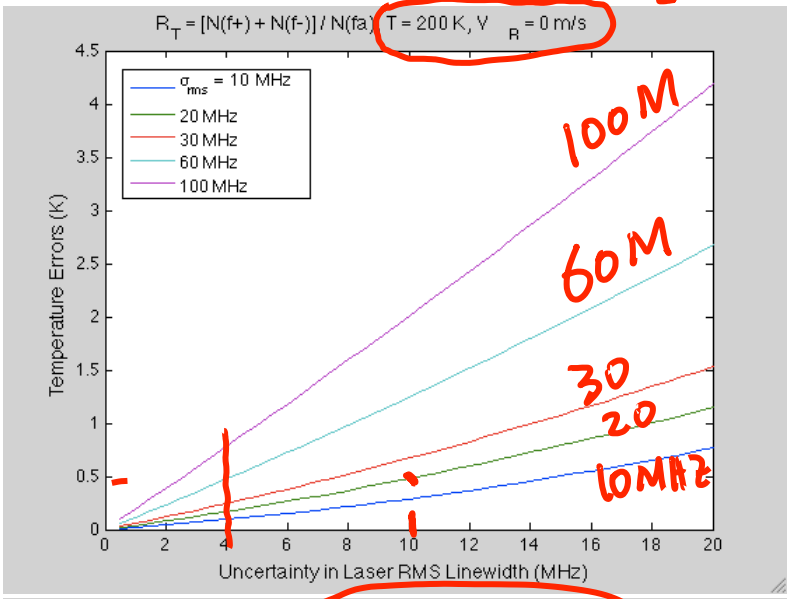
➢This differentiation approach is a method generally applicable for lidars using ratio techniques, not only Na Dopper lidar, but also Fe and K Doppler lidars, and others like edge-filter technique wind lidars, etc.

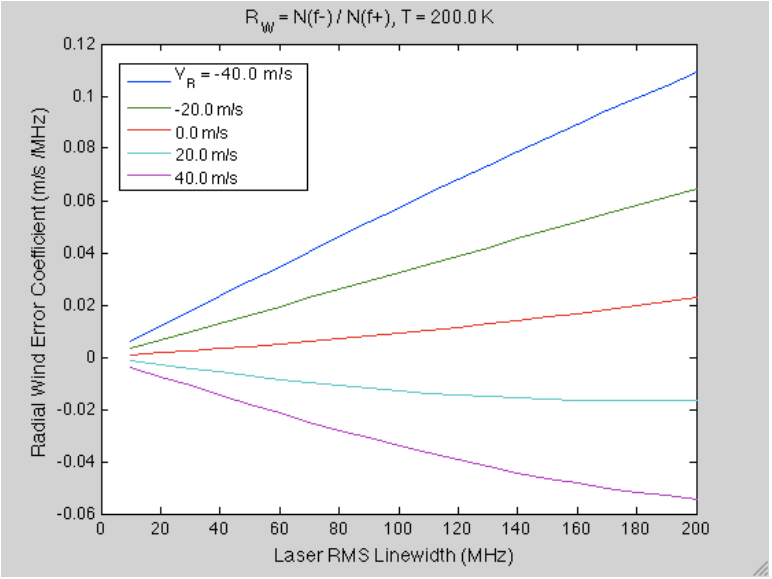


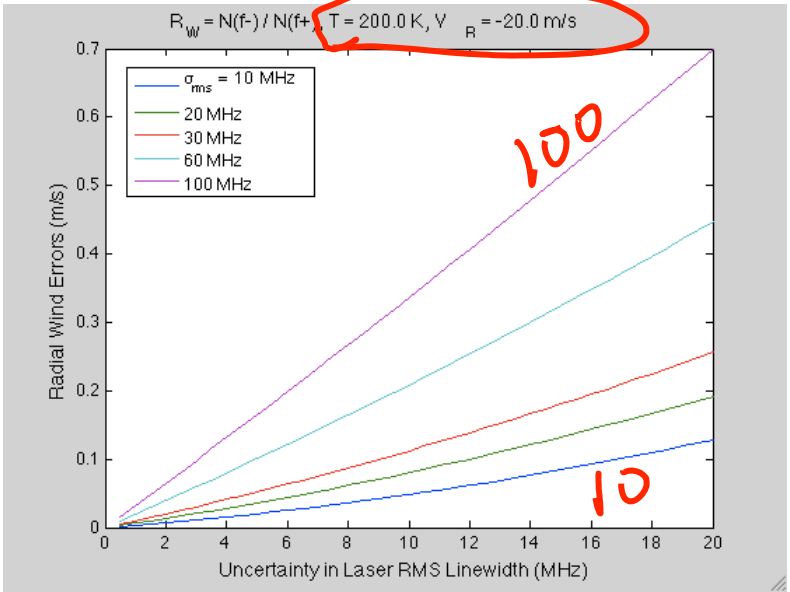
Example Results for 3-Freq Na Lidar: Laser Linewidth Influence

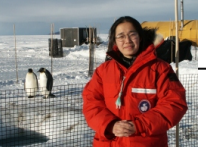
σ_L









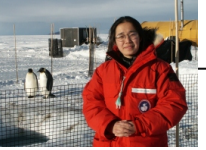


Laser Linewidth and Uncertainty

- When the laser rms linewidth (σ_{rms}) is smaller, the temperature and wind errors caused by the same uncertainty in laser linewidth are smaller.

- For 60 MHz rms linewidth (like the current dye-laser-based Na Doppler lidar), 4 MHz rms width uncertainty is acceptable.

- If the solid-state Na Doppler lidar has laser rms linewidth to about 30 MHz, then the acceptable rms width uncertainty can be larger.



Laser Linewidth vs. Temp Error

$$\sigma_e^2 = \sigma_D^2 + \sigma_L^2 \longleftarrow \sigma_{rms}$$

$$\downarrow$$

$$\frac{k_B T}{M\lambda^2}$$

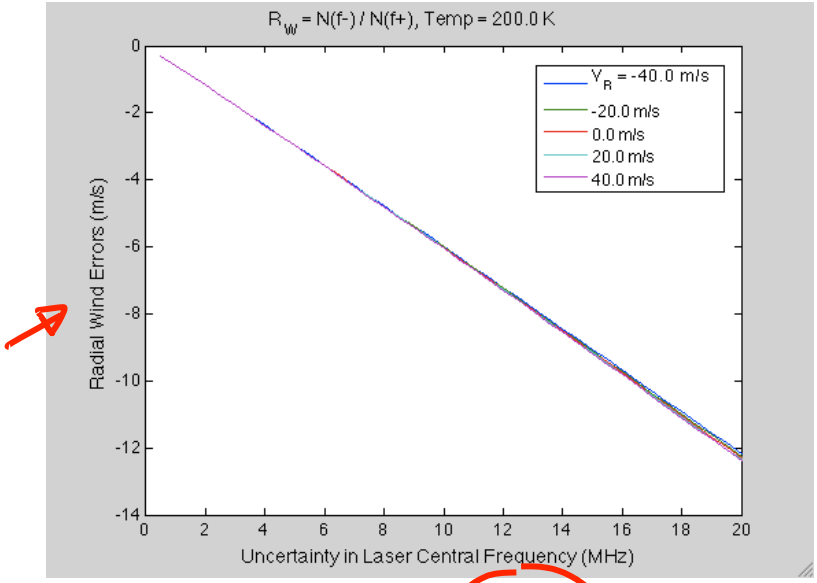
$$\Rightarrow \sigma_D = \sqrt{\sigma_e^2 - \sigma_L^2}$$

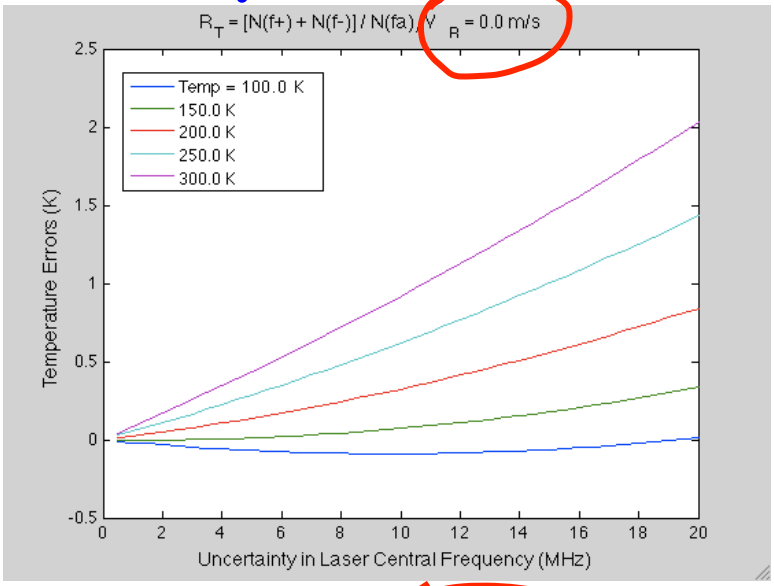
$$= \sqrt{\frac{k_B T}{M\lambda^2}}$$

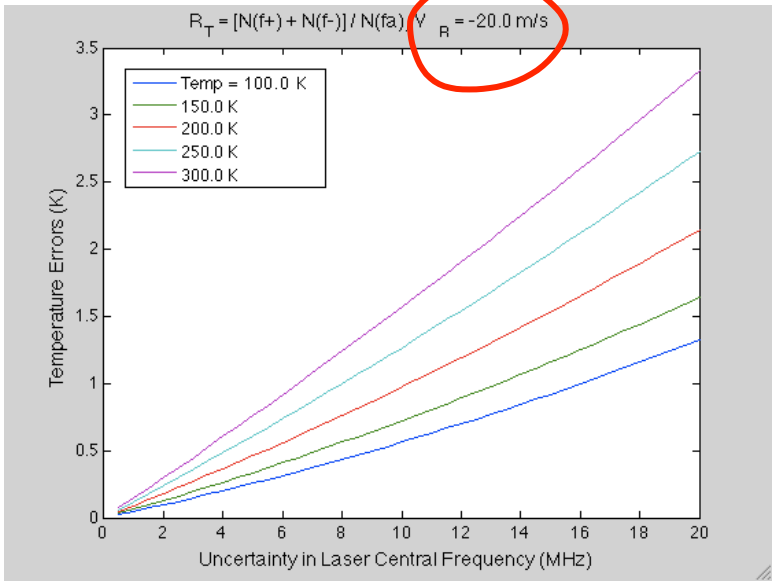
$$\frac{\partial \sigma_D}{\partial \sigma_L} = \frac{1}{2}(-2)\sigma_L = -\sigma_L$$

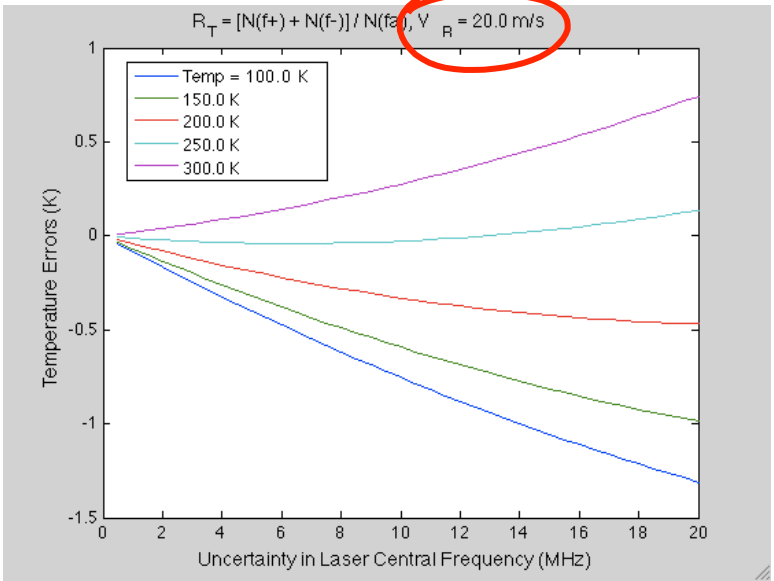
$$\Rightarrow \Delta\sigma_D = -\sigma_L \cdot \Delta\sigma_L$$

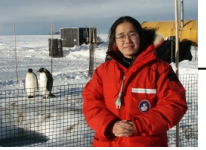
Example Results for 3-Freq Na Lidar: Laser Central Frequency











Laser Central Frequency including chirp and jitter

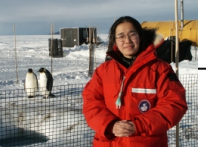
- Wind errors are much more sensitive to the uncertainty or bias in the laser central frequency than temperature errors.

- To keep less than 1 K temperature error, 10 MHz uncertainty or bias in laser central frequency is acceptable; however, 10 MHz would result in about 6 m/s wind error.

- To keep less than 1 m/s wind error, the uncertainty or bias in laser central frequency should be less than 2 MHz.

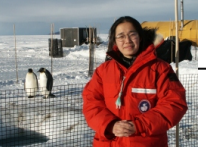
$f_{shift} = \nu_L \cdot \frac{2v}{c} \Rightarrow \Delta f_{shift} = \frac{2v}{c} \Delta \nu_L$

$\Rightarrow \Delta v$? $\Delta \nu_L$



Monte Carlo Method

- To reveal how random error sources affect the measurement precision and accuracy, an approach different than the above analytical "differentiation method" is the "Monte Carlo Method".
- It is not easy to repeat lidar observations in reality, but it is definitely achievable in lidar simulation and error analysis. The Monte Carlo method is to repeat the simulation many times with random sampling of the interested lidar or atmospheric or atomic parameters within their random error ranges and then check how the measurement results are deviated from the true values.
- For example, the laser central frequency has random errors due to frequency jitter. To investigate how it affects the measurements, we may run the simulation of single shot many times and for each shot we let the laser central frequency randomly pick one value within its jitter range. By integrating many shots together, we then look at how the temperature or wind ratios are deviated from the expected ratios if all the shots have the accurate laser frequency.



Summary

- Lidar simulation, error and sensitivity analyses are the "lidar modeling". It is an integration of complicated lidar remote sensing procedure. Error and sensitivity analysis is an important part for lidar research. One approach is to use the "differentiation method", and another one is the Monte Carlo approach.
- The key is still our understanding of the lidar theory and the physical interactions between the laser light and the objects you want to study. Only when we clearly understand the interactions in the atmosphere and the entire lidar detection procedure could we do good lidar simulation and error analysis.
- Accuracy and precision are two different concepts for lidar error analysis. Accuracy concerns about bias, usually determined by systematic errors. Precision concerns about uncertainty, mainly determined by random errors, and in lidar photon counting case, mainly by photon noise.
- The differentiation of metric ratio method can apply to both systematic and random errors, depending on the nature of the errors. Error sources could be systematic bias or random jitter, and measurement errors could also be systematic or random errors. It also works for both error analysis and sensitivity analysis.