# Lecture 08. Fundamentals of Lidar Remote Sensing (4) "Solutions for Lidar Equations"

- Lidar Classification with Physical Processes
- Solution for scattering form lidar equation
- Solution for Raman lidar equation
- Solution for differential absorption lidar equation
- Solution for fluorescence form lidar equation
- Solution for resonance fluorescence lidar
- Solution for Rayleigh and Mie lidar in middle atmos
- Summary

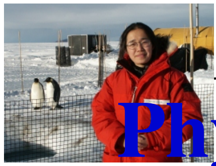


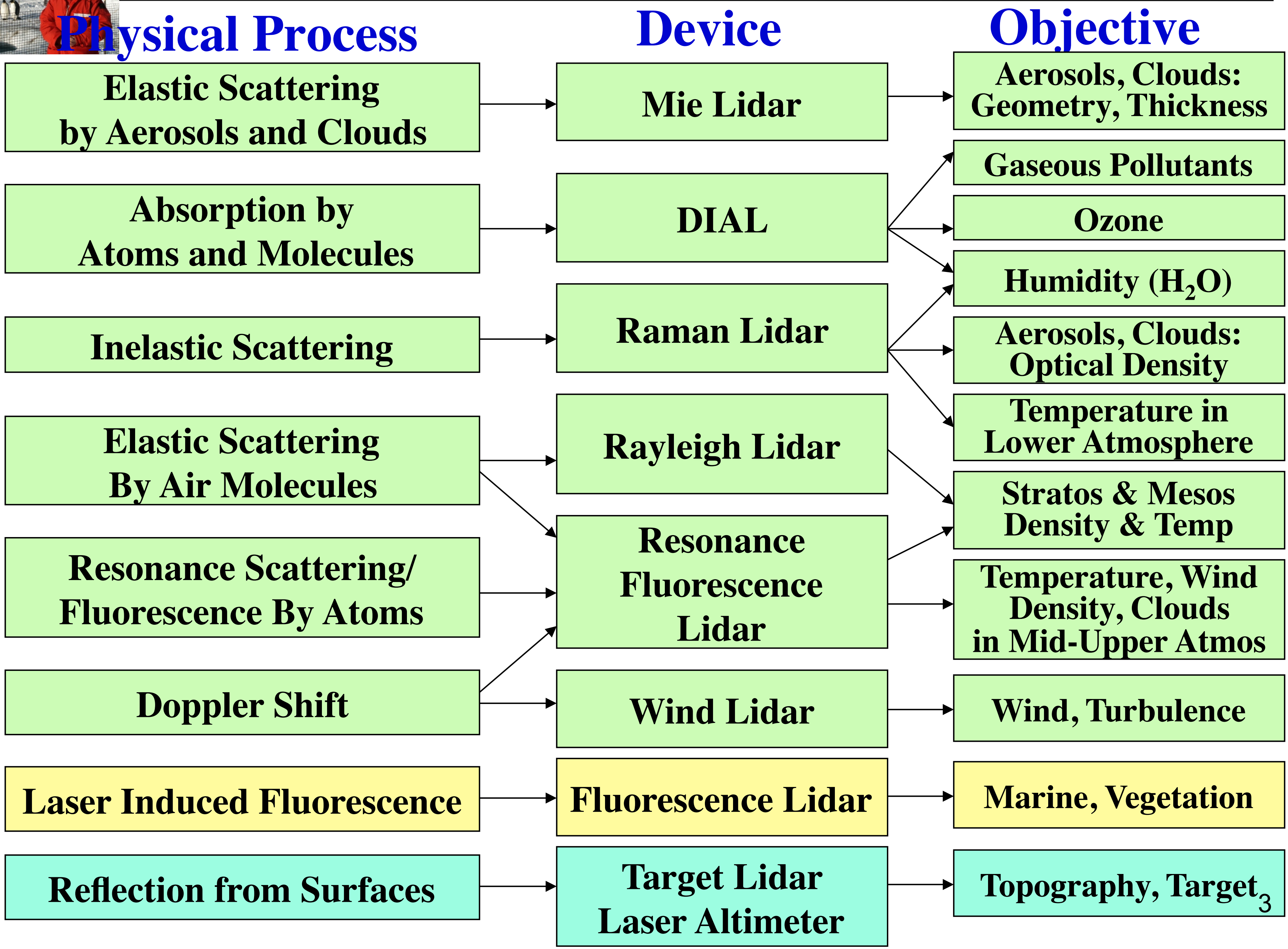
# Review Lidar Equation

- Lidar equation is to link the expected lidar returns ($N_S$) to the lidar parameters (both transmitter and receiver), transmission through medium, physical interactions between light and objects, and background/noise conditions, etc.

- Keep in mind the big picture of a lidar system –
  - Radiation source
  - Radiation propagation in the medium
  - Interaction of radiation with the objects
  - Signal propagation in the medium
  - Photons are collected, filtered and detected

Can you derive a lidar equation by yourself?




Physical Process
Device
Objective
Elastic Scattering by Aerosols and Clouds
Mie Lidar
Aerosols, Clouds: Geometry, Thickness
Absorption by Atoms and Molecules
DIAL
Gaseous Pollutants
Ozone
Humidity ($H_2O$)
Inelastic Scattering
Raman Lidar
Aerosols, Clouds: Optical Density
Temperature in Lower Atmosphere
Elastic Scattering By Air Molecules
Rayleigh Lidar
Stratos & Mesos Density & Temp
Resonance Scattering/ Fluorescence By Atoms
Resonance Fluorescence Lidar
Temperature, Wind Density, Clouds in Mid-Upper Atmos
Doppler Shift
Wind Lidar
Wind, Turbulence
Laser Induced Fluorescence
Fluorescence Lidar
Marine, Vegetation
Reflection from Surfaces
Target Lidar Laser Altimeter
Topography, Target



# Parameters in General Lidar Equation

Assumptions: independent and single scattering

$$N_S(\lambda,R)=\left[\frac{P_L(\lambda_L)\Delta t}{hc/\lambda_L}\right]\cdot\left[\beta(\lambda,\lambda_L,\theta,R)\Delta R\right]\cdot\frac{A}{R^2}\cdot\left[T(\lambda_L,R)T(\lambda,R)\right]\cdot\left[\eta(\lambda,\lambda_L)G(R)\right]+N_B \tag{8.1}$$

$N_S$ (R) - expected received photon number from a distance R

$P_L$ - transmitted laser power, $\lambda_L$ - laser wavelength

$\Delta$t - integration time,

h - Planck's constant, c - light speed

$\beta$(R) - volume scatter coefficient at distance R for angle $\theta$,

$\Delta$R - thickness of the range bin

A - area of receiver,

T(R) - one way transmission of the light from laser source to distance R or from distance R to the receiver,

$\eta$ - system optical efficiency,

G(R) - geometrical factor of the system,

$N_B$ - background and detector noise photon counts.



# Review Lidar Equation

- General lidar equation with angular scattering coefficient

$$N_S(\lambda,R) = N_L(\lambda_L)\cdot\left[\beta(\lambda,\lambda_L,\theta,R)\Delta R\right]\cdot\frac{A}{R^2}\cdot\left[T(\lambda_L,R)T(\lambda,R)\right]\cdot\left[\eta(\lambda,\lambda_L)G(R)\right]+N_B \quad (4.2)$$

- General lidar equation with total scattering coefficient

$$N_S(\lambda,R) = N_L(\lambda_L)\cdot\left[\beta_T(\lambda,\lambda_L,R)\Delta R\right]\cdot\frac{A}{4\pi R^2}\cdot\left[T(\lambda_L,R)T(\lambda,R)\right]\cdot\left[\eta(\lambda,\lambda_L)G(R)\right]+N_B \quad (4.12)$$

- General lidar equation in angular scattering coefficient β and extinction coefficient α form

$$N_S(\lambda,R) = \left[\frac{P_L(\lambda_L)\Delta t}{hc/\lambda_L}\right]\left[\beta(\lambda,\lambda_L,\theta,R)\Delta R\right]\left(\frac{A}{R^2}\right) \quad (4.17)$$

$$\cdot\exp\left[-\int_0^R \alpha(\lambda_L,r')dr'\right]\exp\left[-\int_0^R \alpha(\lambda,r')dr'\right]\left[\eta(\lambda,\lambda_L)G(R)\right]+N_B$$



# Specific Lidar Equations

- Lidar equation for Rayleigh lidar

$$N_S(\lambda,R) = \left(\frac{P_L(\lambda)\Delta t}{hc/\lambda}\right)\left(\beta(\lambda,R)\Delta R\right)\left(\frac{A}{R^2}\right)T^2(\lambda,R)\left(\eta(\lambda)G(R)\right) + N_B \quad (5.17)$$

- Lidar equation for resonance fluorescence lidar

$$N_S(\lambda,R) = \left(\frac{P_L(\lambda)\Delta t}{hc/\lambda}\right)\left(\sigma_{eff}(\lambda,R)n_c(z)R_B(\lambda)\Delta R\right)\left(\frac{A}{4\pi R^2}\right)\left(T_a^2(\lambda,R)T_c^2(\lambda,R)\right)\left(\eta(\lambda)G(R)\right) + N_B$$

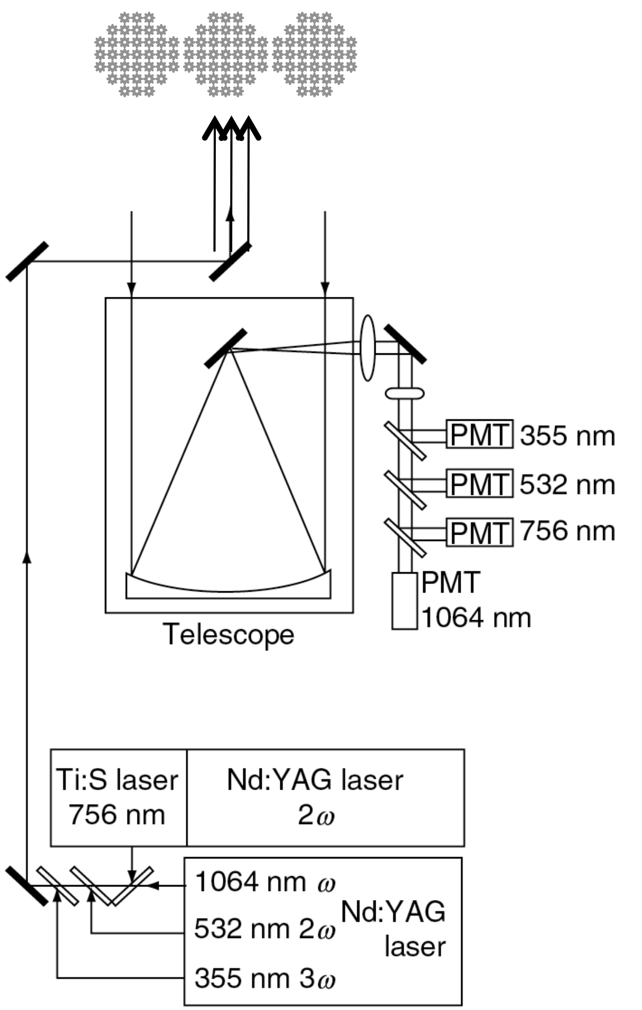
(4.14)

- Lidar equation for differential absorption lidar

$$N_S(\lambda_{on}^{off},R) = N_L(\lambda_{on}^{off})\left[\beta_{sca}(\lambda_{on}^{off},R)\Delta R\right]\left(\frac{A}{R^2}\right)\exp\left[-2\int_0^z \overline{\alpha}(\lambda_{on}^{off},r')dr'\right]$$
$$\times \exp\left[-2\int_0^z \sigma_{abs}(\lambda_{on}^{off},r')n_c(r')dr'\right]\left[\eta(\lambda_{on}^{off})G(R)\right] + N_B \quad (5.21\text{-}5.22)$$

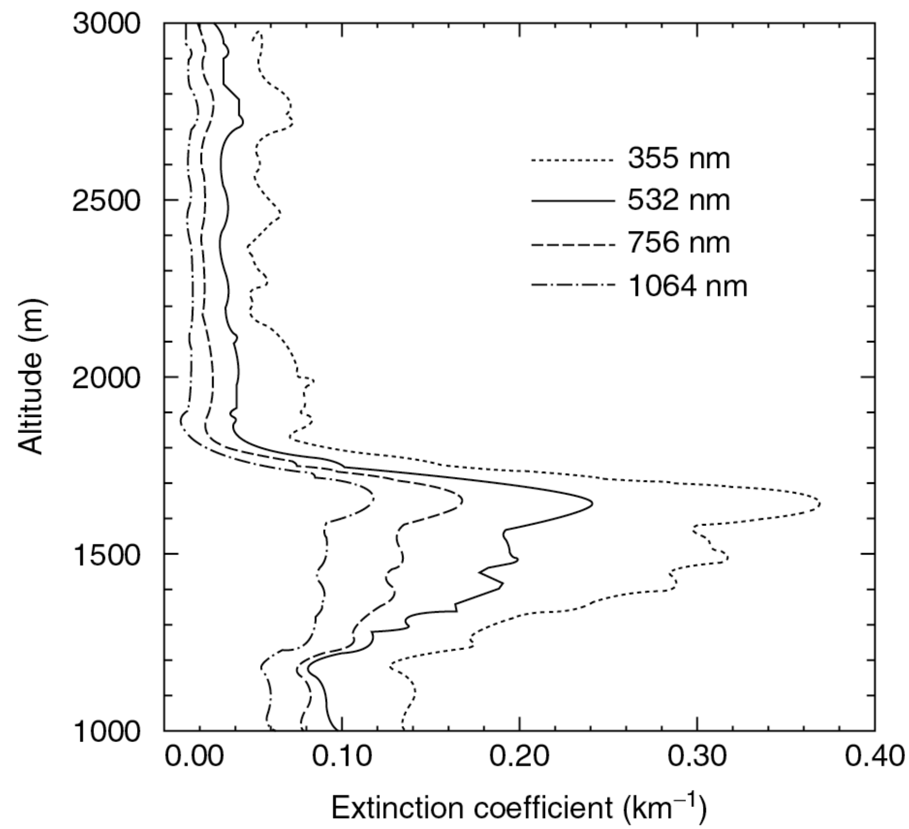


# Elastic Scattering Lidar: Rayleigh Lidar and Mie Lidar




Block diagram of four-wavelength lidar.

## Strong Mie Scattering above Rayleigh Background




Chapter 3 in our textbook



# Solution for Scattering Form Lidar Equation

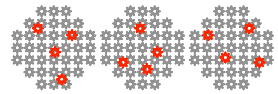
- Scattering form lidar equation

$$N_S(\lambda,R) = \left[\frac{P_L(\lambda_L)\Delta t}{hc/\lambda_L}\right]\cdot\left[\beta(\lambda,\lambda_L,R)\Delta R\right]\cdot\left[\frac{A}{R^2}\right]\cdot\left[T(\lambda_L,R)T(\lambda,R)\right]\cdot\left[\eta(\lambda,\lambda_L)G(R)\right] + N_B \quad (8.2)$$

- Solution for scattering form lidar equation

$$\beta(\lambda,\lambda_L,R) = \frac{N_S(\lambda,R) - N_B}{\left[\frac{P_L(\lambda_L)\Delta t}{hc/\lambda_L}\right]\Delta R\left(\frac{A}{R^2}\right)\left[T(\lambda_L,R)T(\lambda,R)\right]\left[\eta(\lambda,\lambda_L)G(R)\right]} \quad (8.3)$$

# Inelastic Scattering Lidar: Raman Lidar



$H_2O$ in atmosphere

Raman spectra for $O_2$, $N_2$ and $H_2O$ under 355nm laser wavelength

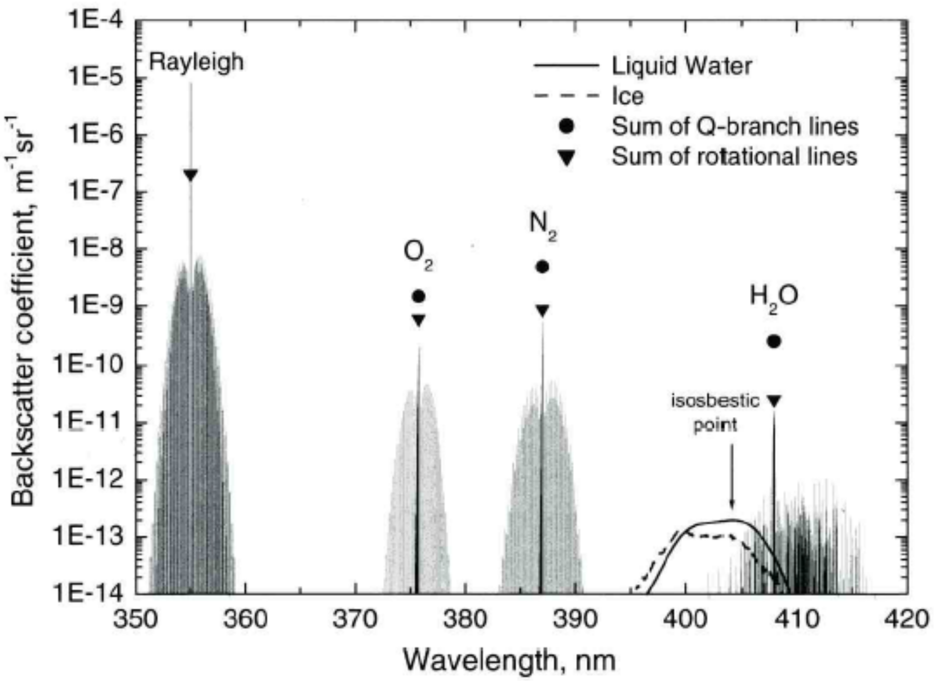



**Fig. 9.2.** Raman backscatter spectrum of the atmosphere for an incident laser wavelength of 355 nm, normal pressure, a temperature of 300 K, an $N_2$ and $O_2$ content of 0.781 and 0.209, respectively, and a water-vapor mixing ratio of 10 g/kg. The curves for liquid water and ice are arbitrarily scaled. The isosbestic point is discussed in Subsection 9.5.2.

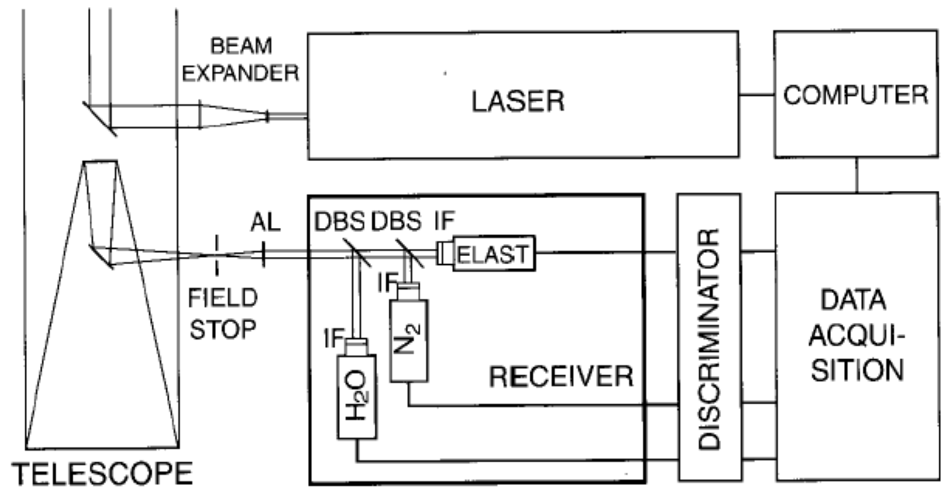



**Fig. 9.3.** Typical setup of a water-vapor and aerosol Raman lidar. AL – achromatic lens, IF – interference filter, DBS – dichroic beam splitter.



# Solution for Raman Lidar Equation

- Raman lidar equations for gas of interest (8.4) and reference gas (8.5)

$$P_{cRa}(R,\lambda_{cRa}) = \frac{E_o \eta_{\lambda_{cRa}}}{R^2} O(R,\lambda_{cRa}) \beta_{cRa}(R,\lambda_0,\lambda_{cRa}) \exp\left\{-\int_0^R [\alpha(r,\lambda_0) + \alpha(r,\lambda_{cRa})] dr\right\} \quad (8.4)$$

$$P_{RefRa}(R,\lambda_{RefRa}) = \frac{E_o \eta_{\lambda_{RefRa}}}{R^2} O(R,\lambda_{RefRa}) \beta_{RefRa}(R,\lambda_0,\lambda_{RefRa}) \exp\left\{-\int_0^R [\alpha(r,\lambda_0) + \alpha(r,\lambda_{RefRa})] dr\right\} \quad (8.5)$$

- Solution for Raman lidar equations

$$\frac{P_{cRa}(R,\lambda_{cRa})}{P_{RefRa}(R,\lambda_{RefRa})} = \frac{\eta_{\lambda_{cRa}}}{\eta_{\lambda_{RefRa}}} \frac{N_{cRa}(R) \frac{d\sigma_{cRa}}{d\Omega}(\pi,\lambda_0,\lambda_{cRa})}{N_{RefRa}(R) \frac{d\sigma_{RefRa}}{d\Omega}(\pi,\lambda_0,\lambda_{RefRa})} \frac{\exp\left\{-\int_0^R \alpha(r,\lambda_{cRa}) dr\right\}}{\exp\left\{-\int_0^R \alpha(r,\lambda_{RefRa}) dr\right\}}$$

$$Mixing \quad Ratio = \frac{N_{cRa}(R)}{N_{RefRa}(R)} \quad (8.6)$$



# Differential Absorption Lidar (DIAL)

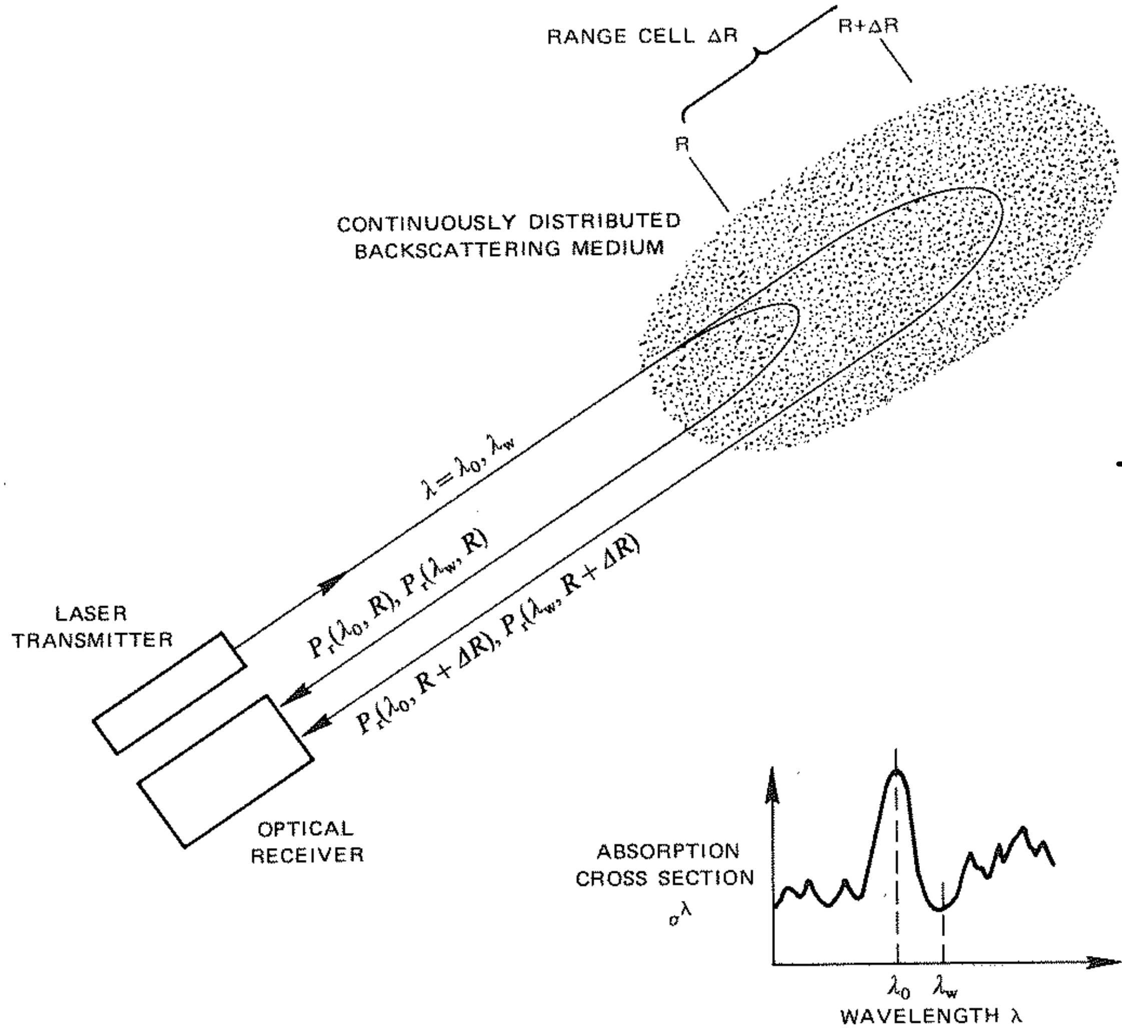



Fig. 4.10. The differential absorption lidar technique

This is a nice schematic, but in reality, light cannot turn as drawn!

Collis and Russell in "Laser monitoring of the atmosphere"



# Differential Absorption/Scattering Form

- For the laser with wavelength $\lambda_{on}$ on the molecular absorption line

$$N_S(\lambda_{on},R) = N_L(\lambda_{on})[\beta_{sca}(\lambda_{on},R)\Delta R]\left(\frac{A}{R^2}\right)\exp\left[-2\int_0^R \overline{\alpha}(\lambda_{on},r')dr'\right] \times \exp\left[-2\int_0^R \sigma_{abs}(\lambda_{on},r')n_c(r')dr'\right]\left[\eta(\lambda_{on})G(R)\right] + N_B \quad (8.7)$$

- For the laser with wavelength $\lambda_{off}$ off the molecular absorption line

$$N_S(\lambda_{off},R) = N_L(\lambda_{off})\left[\beta_{sca}(\lambda_{off},R)\Delta R\right]\left(\frac{A}{R^2}\right)\exp\left[-2\int_0^R \overline{\alpha}(\lambda_{off},r')dr'\right] \times \exp\left[-2\int_0^R \sigma_{abs}(\lambda_{off},r')n_c(r')dr'\right]\left[\eta(\lambda_{off})G(R)\right] + N_B \quad (8.8)$$



# Differential Absorption/Scattering Form

- The ratio of photon counts from these two channels is a function of the differential absorption and scattering:

$$\frac{N_S(\lambda_{on},R)-N_B}{N_S(\lambda_{off},R)-N_B}=\frac{N_L(\lambda_{on})\beta_{sca}(\lambda_{on},R)}{N_L(\lambda_{off})\beta_{sca}(\lambda_{off},R)}\frac{\eta(\lambda_{on})}{\eta(\lambda_{off})}$$
$$\times\exp\left\{-2\int_0^R\left[\bar{\alpha}(\lambda_{on},r')-\bar{\alpha}(\lambda_{off},r')\right]dr'\right\}$$
$$\times\exp\left\{-2\int_0^R\left[\sigma_{abs}(\lambda_{on},r')-\sigma_{abs}(\lambda_{off},r')\right]n_c(r')dr'\right\} \quad (8.9)$$

$$\Delta\sigma=\sigma_{abs}(\lambda_{on})-\sigma_{abs}(\lambda_{off}) \quad (8.10)$$



# Solution for Differential Absorption Lidar Equation

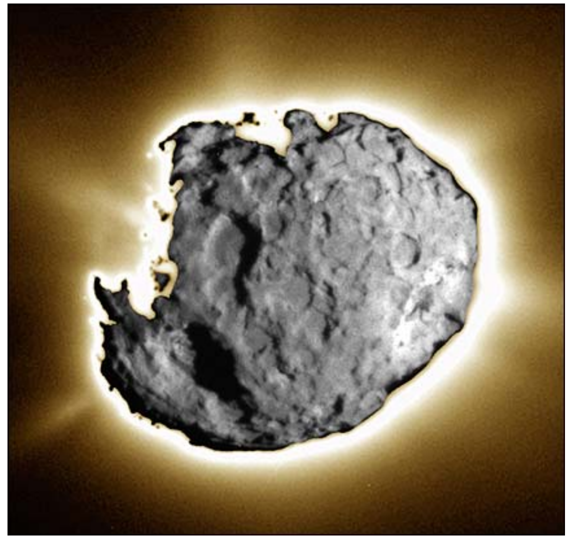
- Solution for differential absorption lidar equation

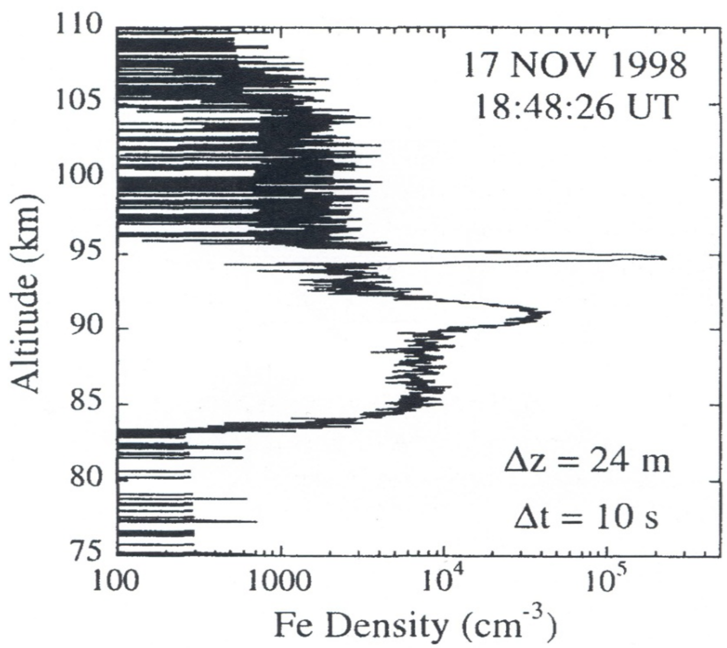
$$n_c(R) = \frac{1}{2\Delta\sigma}\frac{d}{dR}\left\{\begin{array}{l} \ln\left[\dfrac{N_L(\lambda_{on})\beta_{sca}(\lambda_{on},R)}{N_L(\lambda_{off})\beta_{sca}(\lambda_{off},R)}\dfrac{\eta(\lambda_{on})}{\eta(\lambda_{off})}\right] \\ -\ln\left[\dfrac{N_S(\lambda_{on},R)-N_B}{N_S(\lambda_{off},R)-N_B}\right] \\ -2\int_0^R\left[\overline{\alpha}(\lambda_{on},r')-\overline{\alpha}(\lambda_{off},r')\right]dr' \end{array}\right\} \quad (8.11)$$

$$\Delta\sigma = \sigma_{abs}(\lambda_{on}) - \sigma_{abs}(\lambda_{off}) \quad (8.10)$$

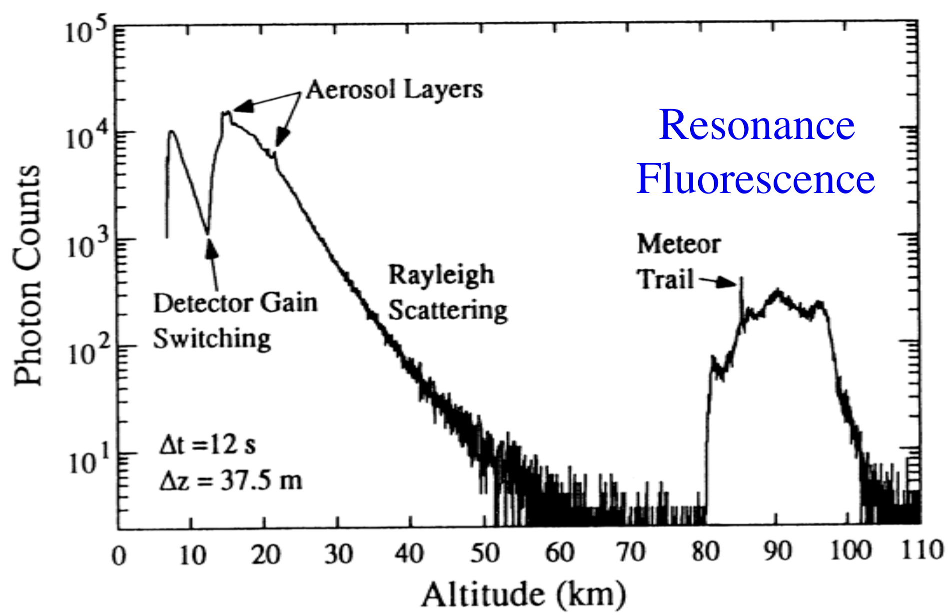


# Resonance Fluorescence Lidar






Profile on an Fe meteor trail measured by lidar [Chu *et al*., 2000]





# Solution for Fluorescence Form Lidar Equation

- Fluorescence form lidar equation

$$N_S(\lambda,R) = \left(\frac{P_L(\lambda)\Delta t}{hc/\lambda}\right)\left(\sigma_{eff}(\lambda,R)n_c(R)R_B(\lambda)\Delta R\right)\left(\frac{A}{4\pi R^2}\right)\left(T_a^2(\lambda,R)T_c^2(\lambda,R)\right)\left(\eta(\lambda)G(R)\right) + N_B$$

(8.12)

- Solution for fluorescence form lidar equation

$$n_c(R) = \frac{N_S(\lambda,R) - N_B}{\left(\frac{P_L(\lambda)\Delta t}{hc/\lambda}\right)\left(\sigma_{eff}(\lambda)R_B(\lambda)\Delta R\right)\left(\frac{A}{4\pi R^2}\right)\left(\eta(\lambda)T_a{}^2(\lambda,R)T_c^2(\lambda,R)G(R)\right)}$$

(8.13)



# Solution for Resonance Fluorescence Lidar Equation

- Resonance fluorescence and Rayleigh lidar equations

$$N_S(\lambda,z) = \left(\frac{P_L(\lambda)\Delta t}{hc/\lambda}\right)\left(\sigma_{eff}(\lambda,z)n_c(z)R_B(\lambda)\Delta z\right)\left(\frac{A}{4\pi z^2}\right)\left(T_a{}^2(\lambda)T_c^2(\lambda,z)\right)\left(\eta(\lambda)G(z)\right) + N_B \quad (8.12)$$

$$N_R(\lambda,z_R) = \left(\frac{P_L(\lambda)\Delta t}{hc/\lambda}\right)\left(\sigma_R(\pi,\lambda)n_R(z_R)\Delta z\right)\left(\frac{A}{z_R{}^2}\right)T_a{}^2(\lambda,z_R)\left(\eta(\lambda)G(z_R)\right) + N_B \quad (8.14)$$

- Rayleigh normalization

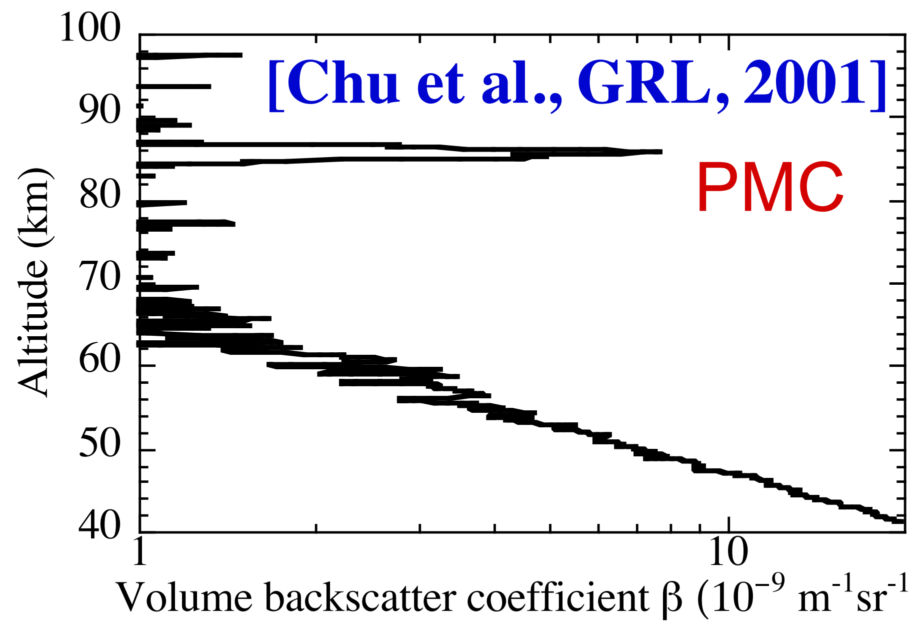
$$\frac{n_c(z)}{n_R(z_R)} = \frac{N_S(\lambda,z)-N_B}{N_R(\lambda,z_R)-N_B}\cdot\frac{z^2}{z_R{}^2}\cdot\frac{4\pi\sigma_R(\pi,\lambda)}{\sigma_{eff}(\lambda,z)R_B(\lambda)}\cdot\frac{\cancel{T_a{}^2(\lambda,z_R)}\cancel{G(z_R)}}{\cancel{T_a{}^2(\lambda,z)}T_c^2(\lambda,z)\cancel{G(z)}} \quad (8.15)$$

- Solution for resonance fluorescence

$$n_c(z) = n_R(z_R)\frac{N_S(\lambda,z)-N_B}{N_R(\lambda,z_R)-N_B}\cdot\frac{z^2}{z_R{}^2}\cdot\frac{4\pi\sigma_R(\pi,\lambda)}{\sigma_{eff}(\lambda,z)R_B(\lambda)}\cdot\frac{1}{T_c^2(\lambda,z)} \quad (8.16)$$



# Rayleigh and Mie Lidar for Middle Atmosphere Detection


Polar mesospheric clouds (PMC), also noctilucent clouds (NLC), are thin scattering layers of nanometer-sized water ice particles, occurring around 80-87 km in the high-latitude summer mesopause region.

Polar mesospheric clouds
Noctilucent clouds

Chapter 5 in our textbook



# Solution for Rayleigh and Mie Lidars

- Rayleigh and Mie (middle atmos) lidar equations

$$N_S(\lambda,z)=\left(\frac{P_L(\lambda)\Delta t}{hc/\lambda}\right)\left(\beta_R(z)+\beta_{aerosol}(z)\right)\Delta z\left(\frac{A}{z^2}\right)T_a^{\,2}(\lambda,z)\left(\eta(\lambda)G(z)\right)+N_B \qquad (8.17)$$

$$N_R(\lambda,z_R)=\left(\frac{P_L(\lambda)\Delta t}{hc/\lambda}\right)\left(\beta_R(z_R)\Delta z\right)\left(\frac{A}{z_R^{\,2}}\right)T_a^{\,2}(\lambda,z_R)\left(\eta(\lambda)G(z_R)\right)+N_B \qquad (8.18)$$

- Rayleigh normalization

$$\frac{\beta_R(z)+\beta_{aerosol}(z)}{\beta_R(z_R)}=\frac{N_S(\lambda,z)-N_B}{N_R(\lambda,z_R)-N_B}\cdot\frac{z^2}{z_R^{\,2}}\cdot\frac{T_a^{\,2}(\lambda,z_R)G(z_R)}{T_a^{\,2}(\lambda,z)G(z)} \qquad (8.19)$$

- For Rayleigh scattering at z and $z_R$

$$\frac{\beta_R(z)}{\beta_R(z_R)}=\frac{\sigma_R(z)n_{atm}(z)}{\sigma_R(z_R)n_{atm}(z_R)}=\frac{n_{atm}(z)}{n_{atm}(z_R)} \qquad (8.20)$$



# Solution (Continued)

- Solution for Mie scattering in middle atmosphere

$$\beta_{aerosol}(z) = \beta_R(z_R)\left[\frac{N_S(\lambda,z)-N_B}{N_R(\lambda,z_R)-N_B}\cdot\frac{z^2}{z_R^2}-\frac{n_{atm}(z)}{n_{atm}(z_R)}\right] \quad (8.21)$$

$$\beta_R(\lambda,z_R,\pi) = 2.938\times 10^{-32}\frac{P(z_R)}{T(z_R)}\cdot\frac{1}{\lambda^{4.0117}}\left(m^{-1}sr^{-1}\right) \quad (5.14)$$

- Rayleigh normalization when aerosols are not present

$$\frac{\beta_R(z)}{\beta_R(z_R)} = \frac{N_S(\lambda,z)-N_B}{N_R(\lambda,z_R)-N_B}\cdot\frac{z^2}{z_R^2}\cdot\frac{\cancel{T_a^2(\lambda,z_R)}\cancel{G(z_R)}}{\cancel{T_a^2(\lambda,z)}\cancel{G(z)}} \quad (8.22)$$

- Solution for relative number density in Rayleigh lidar

$$RND(z) = \frac{n_{atm}(z)}{n_{atm}(z_R)} = \frac{\beta_R(z)}{\beta_R(z_R)} = \frac{N_S(\lambda,z)-N_B}{N_R(\lambda,z_R)-N_B}\cdot\frac{z^2}{z_R^2} \quad (8.23)$$



# Rayleigh Backscatter Cross Section

- It is common in lidar field to calculate the Rayleigh backscatter cross section using the following equation

$$\frac{d\sigma_m(\lambda)}{d\Omega} = 5.45 \cdot \left(\frac{550}{\lambda}\right)^4 \times 10^{-32} \left(m^2 sr^{-1}\right) \quad (5.16)$$

where λ is the wavelength in nm.

- The Rayleigh backscatter cross section can also be estimated from the Rayleigh backscatter coefficient

$$\beta_{Rayleigh}(\lambda, z, \theta = \pi) = 2.938 \times 10^{-32} \frac{P(z)}{T(z)} \cdot \frac{1}{\lambda^{4.0117}} \left(m^{-1} sr^{-1}\right) \quad (5.14)$$

where λ is the wavelength in meter, P in mbar, T in Kelvin.

$$\therefore \frac{d\sigma_m(\lambda)}{d\Omega} = \frac{\beta_{Rayleigh}(\lambda, z, \pi)}{n_{atmos}(z)} \left(m^2 sr^{-1}\right) \quad (8.24)$$



# Summary

- Physical processes in the interaction between radiation and objects classify lidars into different kinds, each possessing unique features and lidar equations along with certain applications.
- Solutions of lidar equations can be obtained by solving the lidar equations directly if all the lidar parameters and atmosphere conditions are well known.
- Solutions for three forms of lidar equations are shown: scattering form, fluorescence form, and differential absorption form, along with a few specific lidar types.
- However, system parameters and atmosphere conditions may vary frequently and are NOT well known to experimenters.
- A good solution is to perform Rayleigh normalization to cancel out most of the system and atmosphere parameters so that the essential and known parts can be solved.