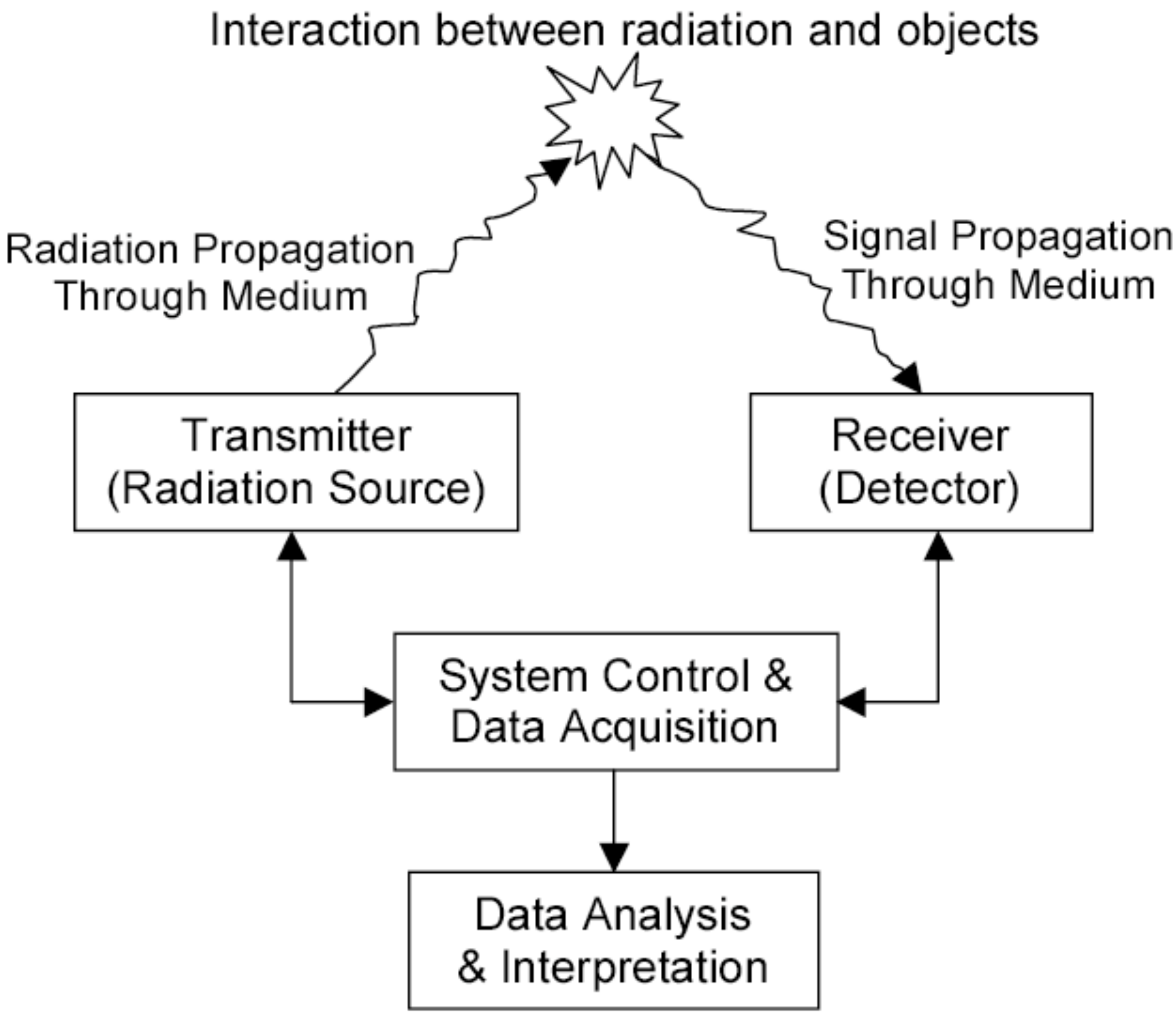
Lecture 11. LIDAR Equation

- Introduction
- Physical processes in lidar remote sensing
- General form of lidar equation
- Consideration of efficiency and geometry
- Scattering form of lidar equation
- Fluorescence form of lidar equation
- Differential absorption/scattering form
- Solutions of lidar equation
- Summary

Introduction

- Lidar equation is a quantitative approach to relate the received photon counts (or light power) with the transmitted laser photon counts (or laser power), light propagation in background atmosphere, physical interaction between light and objects, and lidar system efficiency and geometry, etc.
- The physical processes involved in lidar equation are summarized based on our previously acquired knowledge of QM, spectroscopy, and scattering.
- Lidar equation in different forms is described in great detail with its solutions for different lidars.

Physical Picture in LIDAR Remote Sensing



Physical Processes in LIDAR

- Interaction between light and objects
  - Elastic and inelastic scattering
  - Absorption and differential absorption
  - Fluorescence and resonance fluorescence
  - Doppler shift and Doppler broadening
  - Boltzmann distribution and Extinction
- Light propagation in background atmosphere
  - Extinction = Scattering + Absorption

Physical Processes in LIDAR: Scattering

- Elastic and inelastic scattering
- Elastic scattering in lidar refers to the Rayleigh scattering from molecules and Mie scattering from small particles, in which the particles return to their original states and there is no frequency shift in the scattered light except Doppler effect.
- Inelastic scattering in lidar refers to the Raman scattering from molecules or small particles, in which the particles change their initial states to different final states and the scattered light experience frequency shift due to vibration-rotation Raman shift or pure rotation Raman shift.

Physical Processes in LIDAR: Absorption

- Absorption and differential absorption
- Absorption in lidar (here) mainly refers to the absorption by molecules, but it should also include the absorptions by atoms, small particles, etc.
- Differential absorption in lidar refers to using two wavelengths, one on resonance with an absorption line, while another off resonance. The difference in the absorption cross-sections at these two wavelengths results in the difference between two channels of signals. This is very useful in determining concentration of molecular species.

Physical Processes in LIDAR: Fluorescence

- Resonance fluorescence
- Resonance fluorescence in lidar usually refers to the fluorescence from the metal atoms in the middle and upper atmosphere. Of course, some molecules in upper atmosphere also can give resonance fluorescence, e.g., N_2^+.
- The process contains two steps: the first is for ground-state atoms to absorb incident laser photons, and the second is for the excited atoms to spontaneously emit fluorescence photons.
- This is a first-order process, so has much higher cross-section than nonresonance scattering process.

Physical Processes in LIDAR: Doppler Eff.

- Doppler shift and Doppler broadening
- All atoms and molecules in the atmosphere experience Doppler effect: the Doppler frequency shift dependent on the radial velocity along the laser beam, and the Doppler frequency broadening due to the thermal velocity distribution of atoms and molecules in thermal equilibrium.
- Small particles, like Aerosols, will also have Doppler effect, but due to their very slow velocity, their Doppler effects are usually small enough to be negligible.

Physical Processes in LIDAR

- Boltzmann distribution: in thermal dynamic equilibrium, the populations on different energy levels are determined by the Boltzmann distribution - the higher the energy level, the less the populations.
- Extinction by target scatters: the absorption by atoms and molecules (on resonance) can cause significant extinction to the incident laser power. So for higher altitude bins, we must consider these extinctions by lower altitude atoms and molecules.
- Light extinction in background atmosphere

Light Propagation in LIDAR

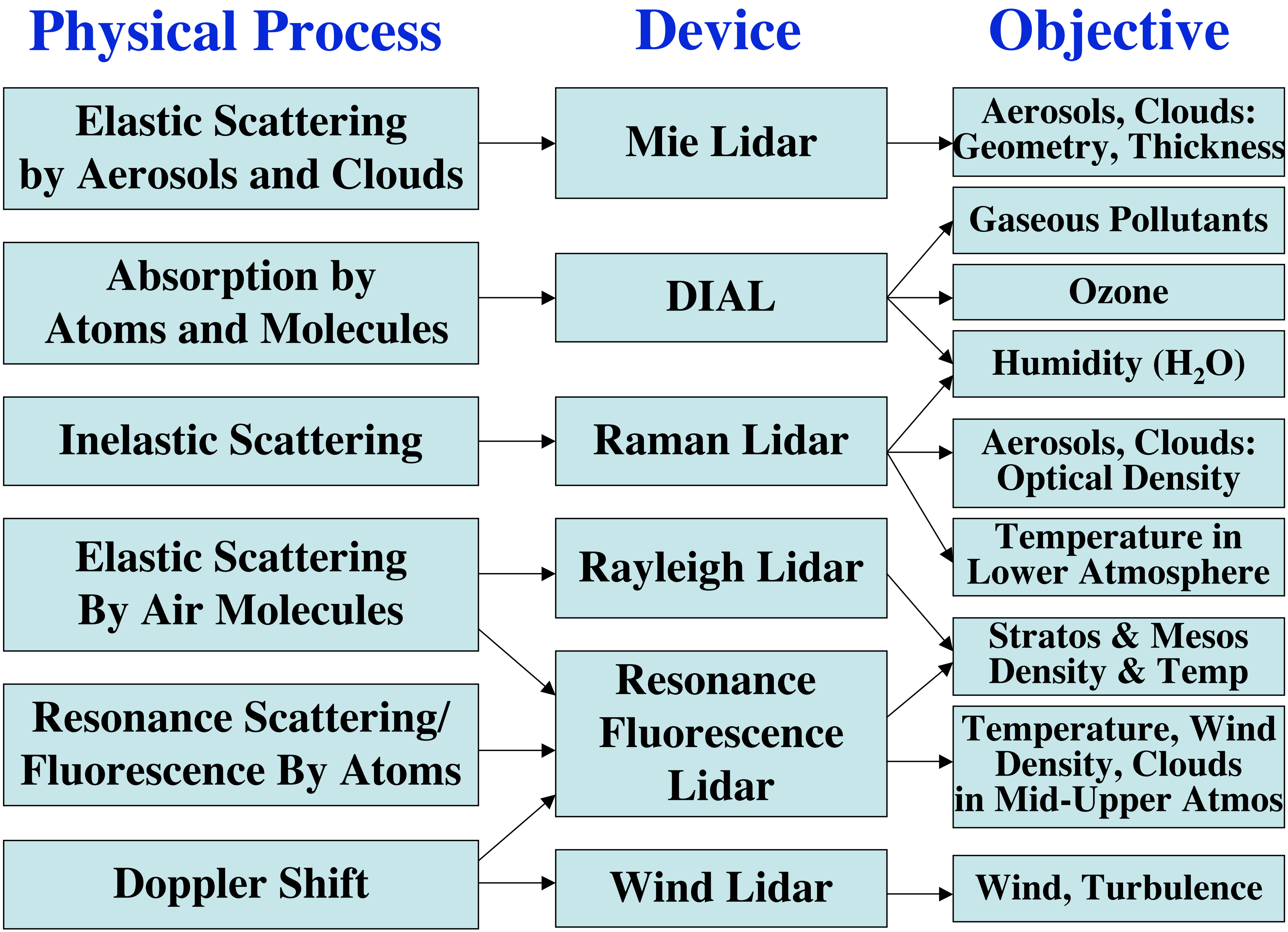
- Light extinction in background atmosphere
- The extinction is mainly caused by two effects: the scattering and absorption by molecules and small particles.

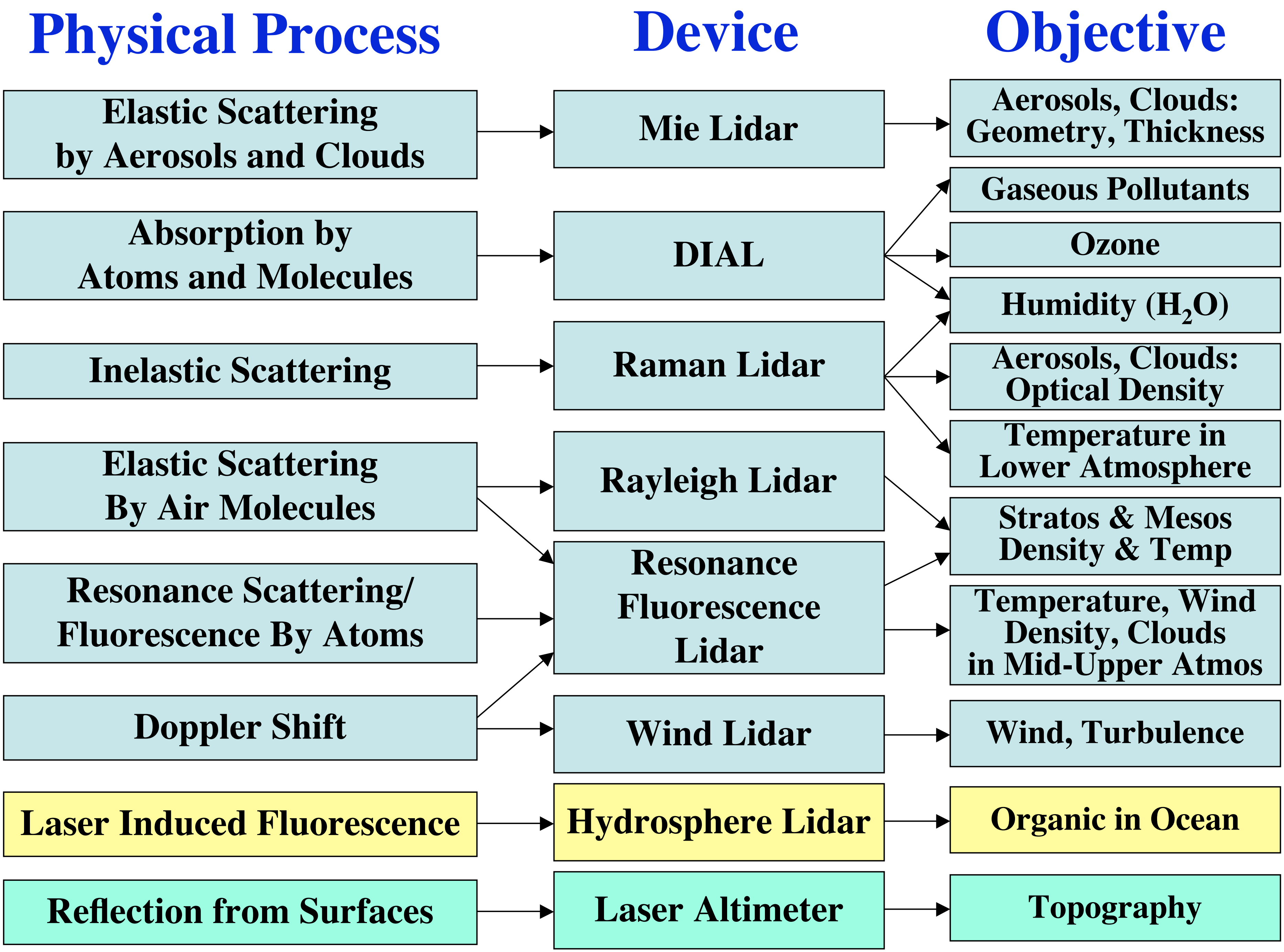
$$\boxed{\alpha(\lambda,R) = \alpha_{mol,sca}(\lambda,R) + \alpha_{mol,abs}(\lambda,R) + \alpha_{aer,sca}(\lambda,R) + \alpha_{aer,abs}(\lambda,R)}$$

$$\boxed{\alpha(\lambda,R) = \sum_i \sigma_{i,ext}(\lambda) N_i(R)}$$

$$\boxed{\sigma_{ext}(\lambda) = \sigma_{sca}(\lambda) + \sigma_{abs}(\lambda)}$$

Integral scattering cross-section over all directions


Physical Process
Device
Objective
Elastic Scattering by Aerosols and Clouds
Mie Lidar
Aerosols, Clouds: Geometry, Thickness
Absorption by Atoms and Molecules
DIAL
Gaseous Pollutants
Ozone
Humidity (H_2O)
Inelastic Scattering
Raman Lidar
Aerosols, Clouds: Optical Density
Temperature in Lower Atmosphere
Elastic Scattering By Air Molecules
Rayleigh Lidar
Stratos & Mesos Density & Temp
Resonance Scattering/ Fluorescence By Atoms
Resonance Fluorescence Lidar
Temperature, Wind Density, Clouds in Mid-Upper Atmos
Doppler Shift
Wind Lidar
Wind, Turbulence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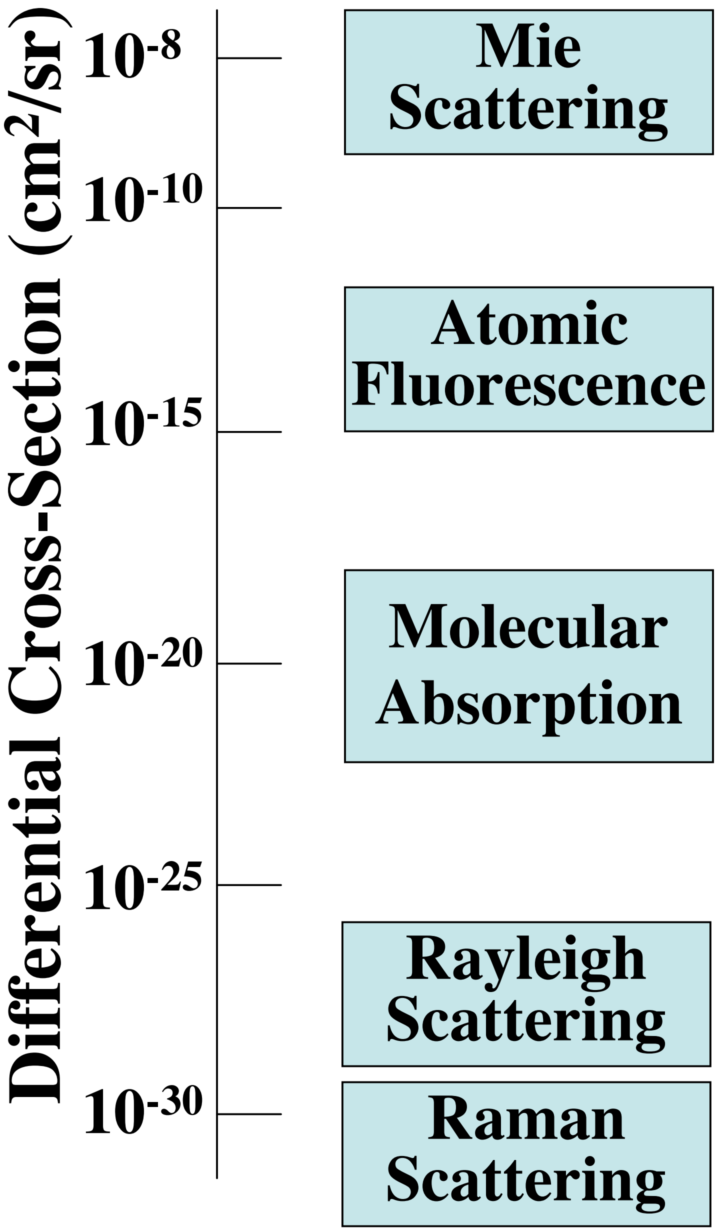
Temperature, Wind Density, Clouds in Mid-Upper Atmos
Doppler Shift
Wind Lidar
Wind, Turbulence
Laser Induced Fluorescence
Hydrosphere Lidar
Organic in Ocean
Reflection from Surfaces
Laser Altimeter
Topography

Comparison of Scattering



$$\boxed{\beta(\lambda,\lambda_L,z) = \sum_i \left[\frac{\mathrm{d}\sigma_i(\lambda_L)}{\mathrm{d}\Omega} n_i(z) p_i(\lambda) \right]}$$

Rayleigh $$\boxed{\frac{d\sigma}{d\Omega} = \sigma_R \propto \frac{1}{\lambda^4}}$$

Resonance Fluorescence

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\sigma_{eff}}{4\pi} = \frac{\int_{-\infty}^{+\infty} \sigma_{abs}(\nu,\nu_0)\cdot g_L(\nu,\nu_L)\,\mathrm{d}\nu}{4\pi}}$$

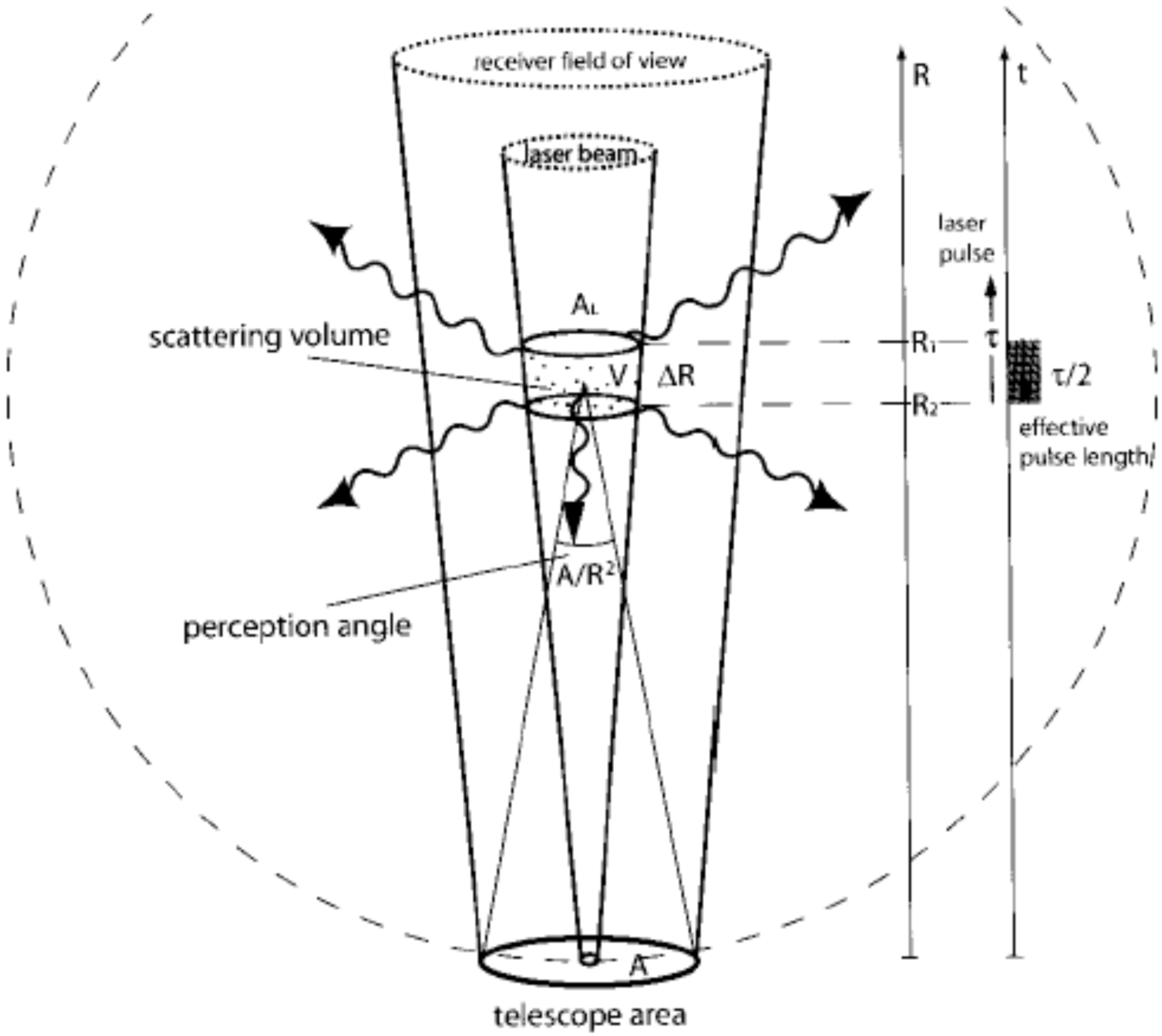
Basic Assumptions for Lidar Equation

- Independent scattering & Single scattering
- Independent scattering: particles are separated adequately and undergo random motion so that the contribution to the total scattered energy by many particles have no phase relation. Thus, the total intensity is simply a sum of the intensity scattered from each particle.
- Single scattering: a photon is scattered only once. Multiple scatter is excluded in our consideration.

General LIDAR Equation

- Lidar equation is the fundamental equation in laser remote sensing field to relate the received photon counts (or light power) with the transmitted laser photon counts (or laser power), light propagation in background atmosphere, physical interaction between light and objects, and lidar system efficiency and geometry, etc.

Illustration for LIDAR Equation



General Form of LIDAR Equation

$$\boxed{N_S(\lambda,R) = N_L(\lambda_L)\cdot\left[\beta(\lambda,\lambda_L,\theta,R)\Delta R\right]\cdot\frac{A}{R^2}\cdot\left[T(\lambda_L,R)T(\lambda,R)\right]\cdot\left[\eta(\lambda,\lambda_L)G(R)\right]+N_B}$$

$$\boxed{P_S(\lambda,R) = P_L(\lambda_L)\cdot\left[\beta(\lambda,\lambda_L,\theta,R)\Delta R\right]\cdot\frac{A}{R^2}\cdot\left[T(\lambda_L,R)T(\lambda,R)\right]\cdot\left[\eta(\lambda,\lambda_L)G(R)\right]+P_B}$$

N_S (R) - expected received photon number from a distance R

N_L - number of transmitted laser photons,

β(R) - volume scatter coefficient at distance R for angle θ,

ΔR - thickness of the range bin

A - area of receiver,

T(R) - one way transmission of the light from laser source to distance R or from distance R to the receiver,

η - system optical efficiency,

G(R) - geometrical factor of the system,

N_B - background photon counts.

LIDAR Equation For Arbitrary Angle

Assumptions: independent and single scattering

$$N_S(\lambda,R) = \left(\frac{P_L(\lambda_L)\Delta t}{hc/\lambda_L}\right)\left(\beta(\lambda,\lambda_L,\theta,R)\Delta R\right)\left(\frac{A}{R^2}\right)\left(T(\lambda_L,R)T(\lambda,R)\right)\left(\eta(\lambda,\lambda_L)G(R)\right) + N_B\Delta t$$

- N_S - expected photon counts detected at λ and distance R;
- 1st term - number of transmitted laser photons;
- 2nd term - probability that a transmitted photon is scattered by the scatters into a unit solid angle at angle θ;
- 3rd term - probability that a scatter photon is collected by the receiving telescope;
- 4th term - light transmission during light propagation from laser source to distance R and from distance R to receiver;
- 5th term - overall system efficiency;
- 6th term - background and detector noise.

LIDAR Equation For Backscattering

Assumptions: independent and single scattering

$$N_S(\lambda,z)=\left(\frac{P_L(\lambda_L)\Delta t}{hc/\lambda_L}\right)\left(\beta(\lambda,\lambda_L,z)\Delta z\right)\left(\frac{A}{z^2}\right)\left(T(\lambda_L,z)T(\lambda,z)\right)\left(\eta(\lambda,\lambda_L)G(z)\right)+N_B\Delta t$$

- N_S - expected photon counts detected at λ and z
- 1st term - number of transmitted laser photons;
- 2nd term - probability that a transmitted photon is backscattered by the scatters into a unit solid angle ($\theta = \pi$);
- 3rd term - probability that a scatter photon is collected by the receiving telescope;
- 4th term - light transmission during light propagation from laser source to distance R and from distance R to receiver;
- 5th term - overall system efficiency;
- 6th term - background and detector noise.

1st Term: Transmitted Photon Number

$$N_S(\lambda,z) = \boxed{\left(\frac{P_L(\lambda_L)\Delta t}{hc/\lambda_L}\right)}\left(\beta(\lambda,\lambda_L,z)\Delta z\right)\left(\frac{A}{z^2}\right)\left(T(\lambda_L,z)T(\lambda,z)\right)\left(\eta(\lambda,\lambda_L)G(z)\right) + N_B\Delta t$$

$$\frac{\text{Laser Power x time bin length}}{\text{Planck constant x Laser frequency}}$$

$$= \frac{\text{Transmitted laser energy within time bin}}{\text{Single laser photon energy}}$$

= Transmitted laser photon number
within time bin length

2nd Term: Probability to be Scattered

$$N_S(\lambda,z) = \left(\frac{P_L(\lambda_L)\Delta t}{hc/\lambda_L}\right)\boxed{\left(\beta(\lambda,\lambda_L,z)\Delta z\right)}\left(\frac{A}{z^2}\right)\left(T(\lambda_L,z)T(\lambda,z)\right)\left(\eta(\lambda,\lambda_L)G(z)\right) + N_B\Delta t$$

Angular scattering probability – the probability that a transmitted photon is backscattered by scatters into a unit solid angle.

Angular scattering probability =
volume backscatter coefficient x scattering layer thickness

Volume backscatter coefficient β is the probability per unit distance travel that a photon is scattered into wavelength λ in unit solid angle at angle $\theta = \pi$.

2nd Term: Probability to be Scattered

$$N_S(\lambda,z) = \left(\frac{P_L(\lambda_L)\Delta t}{hc/\lambda_L}\right)\boxed{\left(\beta(\lambda,\lambda_L,z)\Delta z\right)}\left(\frac{A}{z^2}\right)\left(T(\lambda_L,z)T(\lambda,z)\right)\left(\eta(\lambda,\lambda_L)G(z)\right) + N_B\Delta t$$

Volume backscatter coefficient β is equal to

$$\boxed{\beta(\lambda,\lambda_L,z) = \sum_i \left[\frac{d\sigma_i(\lambda_L)}{d\Omega} n_i(z) p_i(\lambda)\right]}$$

$\dfrac{d\sigma_i(\lambda_L)}{d\Omega}$ Is the differential backscatter cross-section of single particle

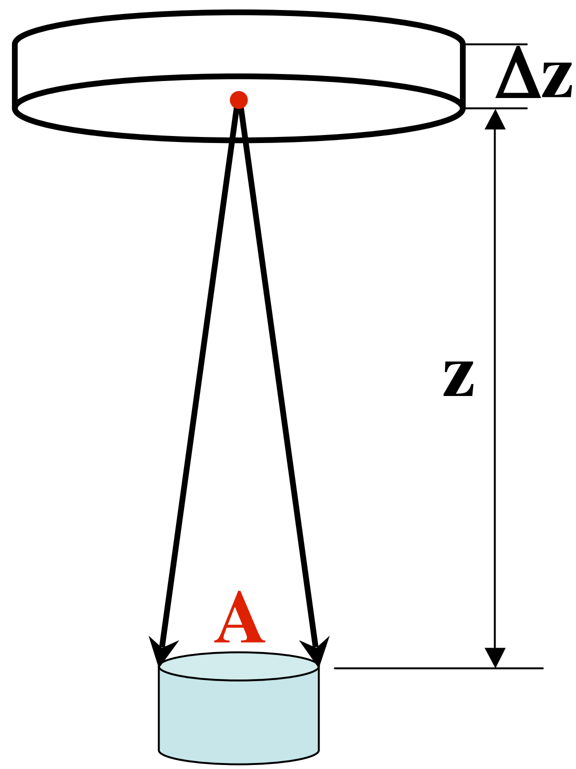
$n_i(z)$ Is the number density of scatter species i

$p_i(\lambda)$ Is the probability of the scattered photons falling into the wavelength λ.

3rd Term: Probability to be Collected

$$N_S(\lambda,z)=\left(\frac{P_L(\lambda_L)\Delta t}{hc/\lambda_L}\right)\left(\beta(\lambda,\lambda_L,z)\Delta z\right)\boxed{\left(\frac{A}{z^2}\right)}\left(T(\lambda_L,z)T(\lambda,z)\right)\left(\eta(\lambda,\lambda_L)G(z)\right)+N_B\Delta t$$

The probability that a scatter photon is collected by the receiving telescope, i.e., the solid angle subtended by the receiver aperture to the scatterer (perception angle)



4th Term: Light Transmission

$$N_S(\lambda,z) = \left(\frac{P_L(\lambda_L)\Delta t}{hc/\lambda_L}\right)\left(\beta(\lambda,\lambda_L,z)\Delta z\right)\left(\frac{A}{z^2}\right)\boxed{\left(T(\lambda_L,z)T(\lambda,z)\right)}\left(\eta(\lambda,\lambda_L)G(z)\right) + N_B\Delta t$$

$\boxed{T(\lambda_L,z)T(\lambda,z)}$ is the atmospheric transmittance at outgoing wavelength λ_L and return wavelength λ

$$T(\lambda_L,R) = \exp\left[-\int_0^R \alpha(\lambda_L,r)dr\right] \qquad T(\lambda,R) = \exp\left[-\int_0^R \alpha(\lambda,r)dr\right]$$

$$\boxed{\begin{aligned} T(\lambda_L,R)T(\lambda,R) &= \exp\left[-\left(\int_0^R \alpha(\lambda_L,r)dr + \int_0^R \alpha(\lambda,r)dr\right)\right] \\ \text{when } \lambda = \lambda_L \quad &= \exp\left[-2\int_0^R \alpha(\lambda,r)dr\right] \end{aligned}}$$

4th Term: Light Transmission

$\alpha(\lambda, R)$ is the extinction coefficient

$$\alpha(\lambda, R) = \sum_i \left[\sigma_{i,ext}(\lambda) n_i(R) \right]$$

$\sigma_{ext}(\lambda)$ is the integral extinction cross-section, since scattering into all directions contributes to light extinction, which is different than the case for backscatter coefficient definition.

$$\sigma_{ext} = \sigma_{scattering} + \sigma_{absorption}$$

5th Term: Overall System Efficiency

$$N_S(\lambda,z) = \left(\frac{P_L(\lambda_L)\Delta t}{hc/\lambda_L}\right)\left(\beta(\lambda,\lambda_L,z)\Delta z\right)\left(\frac{A}{z^2}\right)\left(T(\lambda_L,z)T(\lambda,z)\right)\boxed{\left(\eta(\lambda,\lambda_L)G(z)\right)} + N_B\Delta t$$

$\boxed{\eta(\lambda,\lambda_L) = \eta_T(\lambda_L)\cdot\eta_R(\lambda)}$ is the lidar hardware optical efficiency

e.g., mirrors, lens, filters, detectors, etc

$\boxed{G(z)}$ is the geometrical form factor, mainly concerning the overlap of the area of laser irradiation with the field of view of the receiver optics

6th Term: Background Photon Counts

$$N_S(\lambda,z) = \left(\frac{P_L(\lambda_L)\Delta t}{hc/\lambda_L}\right)\left(\beta(\lambda,\lambda_L,z)\Delta z\right)\left(\frac{A}{z^2}\right)\left(T(\lambda_L,z)T(\lambda,z)\right)\left(\eta(\lambda,\lambda_L)G(z)\right) + \boxed{N_B\Delta t}$$

$\boxed{N_B}$ is the expected photon counts per range bin per unit time, due to background noise (e.g., solar scattering) and detector/circuit shot noise.

General Lidar Equation in β and α

$$N_S(\lambda,z) = \left[\frac{P_L(\lambda_L)\Delta t}{hc/\lambda_L}\right]\left[\beta(\lambda,\lambda_L,z)\Delta z\right]\left(\frac{A}{z^2}\right)\exp\left[-2\int_0^z \alpha(\lambda,z')dz'\right]\left[\eta(\lambda,\lambda_L)G(z)\right] + N_B\Delta t$$

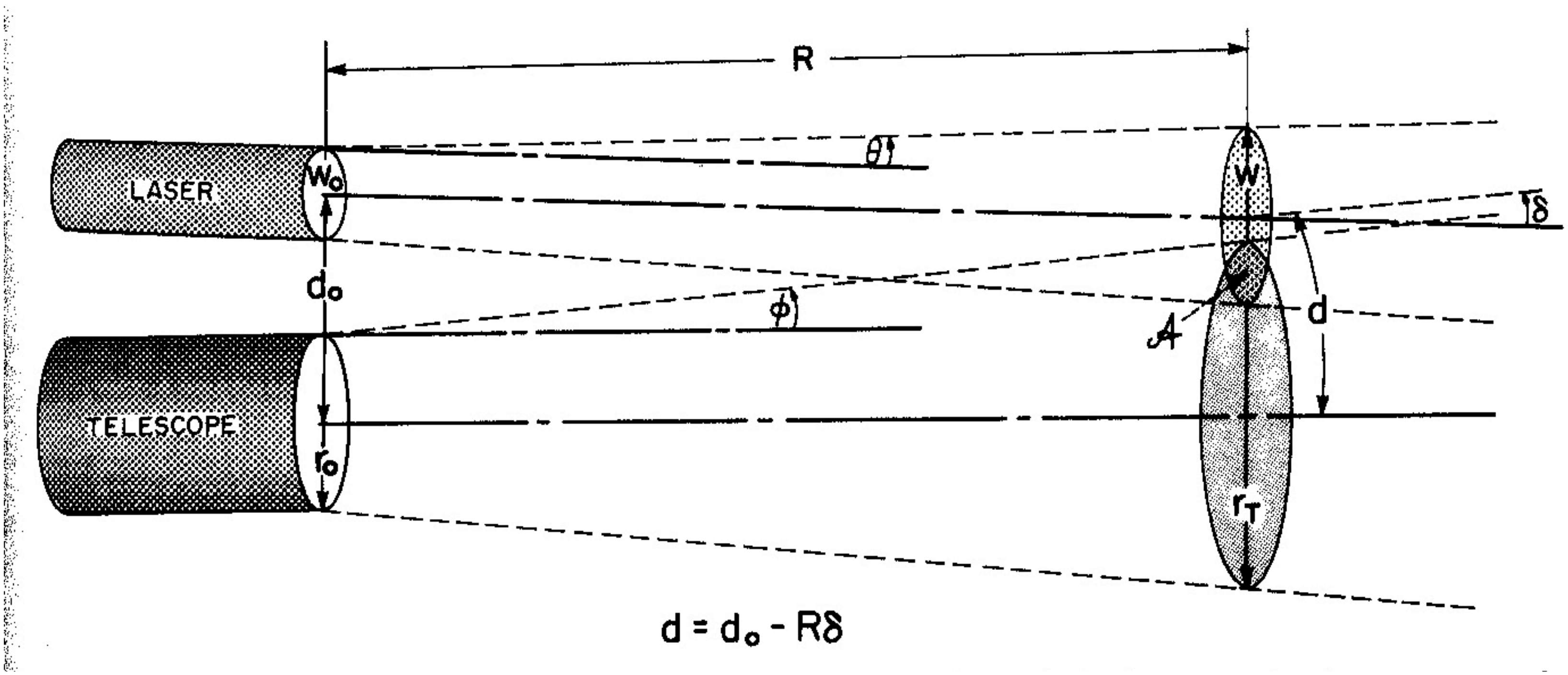
- This form of lidar equation is more generally used by lower atmosphere remote sensing, as it explicitly includes the extinction (scatter + absorption) coefficient, which is easier for Raman, DIAL, and aerosol lidars.

Considerations of Efficiency

$$\eta(\lambda,\lambda_L)=\eta_{opt}(\lambda_L)\eta_{opt}(\lambda)\cdot\eta_{QE}(\lambda)\cdot\eta_{mec}$$

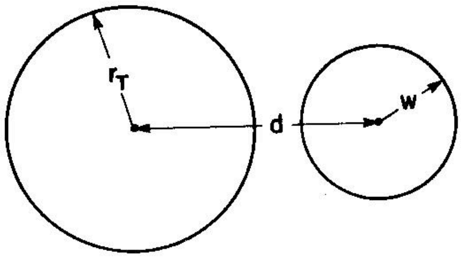
- Optical efficiency
- Quantum efficiency
- Mechanical chopper or gain switching

Considerations of Geometry

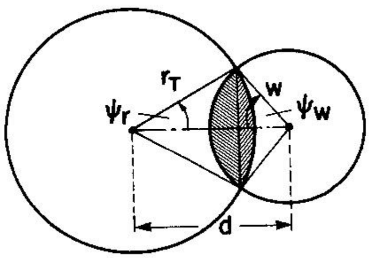


Biaxial lidar geometry

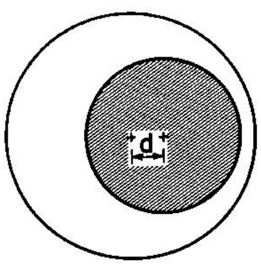
Considerations of Geometry



(a)



(b)



(c)

- ❑ Three overlap situations

Possible for a biaxial lidar

Scattering Form of Lidar Equation For Elastic Scattering

$$N_S(\lambda,z) = \left(\frac{P_L(\lambda)\Delta t}{hc/\lambda}\right)\left(\beta(\lambda,z)\Delta z\right)\left(\frac{A}{z^2}\right)\left(\eta(\lambda)T^2(\lambda,z)G(z)\right) + N_B\Delta t$$

- Rayleigh, Mie, and Raman scattering processes are instantaneous scattering processes, so there are no finite relaxation effects involved, but infinitely short duration.
- For Rayleigh and Mie scattering, there is no frequency shift when the atmospheric particles are at rest, so

$$\lambda = \lambda_L, \quad p_i(\lambda) = 1$$

- The volume backscatter coefficient can be written as

$$\beta(\lambda,z) = \sum_i \left[\frac{d\sigma_i(\lambda)}{d\Omega} n_i(z)\right]$$

Scattering Form of Lidar Equation For Inelastic Scattering

$$N_S(\lambda,z)=\left(\frac{P_L(\lambda_L)\Delta t}{hc/\lambda_L}\right)\left(\beta(\lambda,\lambda_L,z)\Delta z\right)\left(\frac{A}{z^2}\right)\left(T(\lambda_L,z)T(\lambda,z)\right)\left(\eta(\lambda,\lambda_L)G(z)\right)+N_B\Delta t$$

- For Raman scattering, there is large frequency shift and

$$\lambda\neq\lambda_L,\quad p_i(\lambda)\neq 1,\quad p_i(\lambda)<1$$

- The volume backscatter coefficient can be written as

$$\beta(\lambda,\lambda_L,z)=\sum_i\left[\frac{d\sigma_i(\lambda_L)}{d\Omega}n_i(z)p_i(\lambda)\right]$$

Rayleigh Scattering Lidar Equation

- Rayleigh scattering is anisotropic and described by a phase function that is defined as the ratio of energy scattered into a direction per unit angle to the average energy scattered in all directions per unit solid angle.
- Phase function depends on the polarization of the light.
- Goody and Young, Atmospheric Radiation, 1989, give the phase function for natural light (unpolarized light)

$$\boxed{P(\theta) = 0.7629 \times (1 + 0.9324 \cos^2 \theta)}$$

- Volume angular scattering coefficient can be derived from the total scattering coefficient

$$\beta(\theta) = \frac{\beta_T}{4\pi} P(\theta) = \frac{\beta_T}{4\pi} \times 0.7629 \times (1 + 0.9324 \cos^2 \theta)$$

Fluorescence Form of Lidar Equation

$$\boxed{N_S(\lambda,z)=\left(\frac{P_L(\lambda)\Delta t}{hc/\lambda}\right)\left(\sigma_{eff}(\lambda,z)n_c(z)R_B(\lambda)\Delta z\right)\left(\frac{A}{4\pi z^2}\right)\left(T_a^2(\lambda,z)E^2(\lambda,z)\right)\left(\eta(\lambda)G(z)\right)+N_B\Delta t}$$

- Since fluorescence is isotropic, backscatter cross-section can be derived from the total scattering cross-section

$$\boxed{\frac{d\sigma}{d\Omega}=\frac{\sigma_{eff}}{4\pi}}$$

σ_{eff} is the total effective scatter cross-section

- $p_i(\lambda)$ is replaced by the branching ratio R_B, which is defined as the percentage of the fluorescence photons that fall in the absorption resonance wavelength, occupying the entire fluorescence photons emitted by the excited atoms.

- The volume backscatter coefficient can be written as

$$\boxed{\beta(\lambda,\lambda_L,z)=\frac{\sigma_{eff}(\lambda_L,z)}{4\pi}n_c(z)R_B(\lambda)}$$

Fluorescence Form of Lidar Equation

- Resonance fluorescence process involves absorption of an incident photon and spontaneous emission of a fluorescence photon by each scattering atom.
- Compared with the instantaneous scattering processes, a few extra factors must be taken into account for the resonance fluorescence in the middle and upper atmosphere:

1. Laser spectral shape and linewidth
2. Laser and signal polarization
3. Laser and signal extinction due to the absorption of the resonance atomic layers
4. Finite lifetime of the excited state of the atoms
5. Laser pulse duration time and temporal shape
6. Possible stimulated emission
7. Optical pumping effect

Fluorescence Form of Lidar Equation

- The first two factors are considered in the effective backscatter cross-section.
- The third factor is considered as extinction coefficient.
- The last four factors are considered as the saturation and optical pumping effects of the atomic layers.
- The total effective scatter cross-section is determined by the convolution of the atomic absorption cross-section and the laser spectral lineshape.

$$\sigma_{abs}(\nu,\nu_0) = A_{ki}\frac{\lambda^2}{8\pi n^2}\frac{g_k}{g_i}g_A(\nu,\nu_0)$$

Fluorescence Form of Lidar Equation

- The total effective scatter cross-section is determined by the convolution of the atomic absorption cross-section and the laser spectral lineshape.
- Atomic absorption cross-section

$$\sigma_{abs}(\nu,\nu_0) = A_{ki}\frac{\lambda^2}{8\pi n^2}\frac{g_k}{g_i}g_A(\nu,\nu_0)$$

- For typical resonance fluorescence in the middle atmosphere, Doppler broadening of atomic spectral line is much wider than natural linewidth. So the spectral shape g_A can be represented by a Gaussian distribution. Thus,

$$\sigma_{abs}(\nu,\nu_0) = \sigma_o \exp\left\{-\frac{\left[\nu(1-V_R/c)-\nu_0\right]^2}{2\sigma_D^2}\right\}$$

Fluorescence Form of Lidar Equation

- If the laser lineshape can be represented by a Gaussian

$$g_L(\nu,\nu_L) = \frac{1}{\sqrt{2\pi}\sigma_L}\exp\left(-\frac{\left[\nu-\nu_L\right]^2}{2\sigma_L^2}\right)$$

- The total effective cross-section is then given by

$$\sigma_{eff}(\nu_L,\nu_0) = \int_{-\infty}^{+\infty}\sigma_{abs}(\nu,\nu_0)g_L(\nu,\nu_L)d\nu$$

$$= \frac{\sigma_D\sigma_o}{\sqrt{\sigma_D^2+\sigma_L^2}}\exp\left\{-\frac{\left[\nu_L(1-V_R/c)-\nu_0\right]^2}{2\left(\sigma_D^2+\sigma_L^2\right)}\right\}$$

Fluorescence Form of Lidar Equation

- Light transmission in fluorescence case can be written as a product of the transmission in the lower atmosphere T_a and the extinction coefficient E due to the absorption of the atomic layers.

$$\boxed{T(\lambda,z) = T_a(\lambda)E(\lambda,z)}$$

- The extinction is defined as the ratio of the transmitted laser power to the incident laser power at the atomic layers, and can be calculated from the total effective cross-section and atomic density

$$\boxed{E(z) = \exp\left(-\int_{z_{bottom}}^{z} \sigma_{eff}(\lambda,z)n_c(z)\,\mathrm{d}z\right) = \exp\left(-\sum_{z_{bottom}}^{z} \sigma_{eff}(\lambda,z)n_c(z)\Delta z\right)}$$

Fluorescence Form of Lidar Equation

- Saturation and optical pumping effects are complicated. Their equivalent effects are to make corrections to the total effective cross-section.
- If we include the saturation correction factors into the total effective cross-section, then the resonance fluorescence lidar equation can be generally written as

$$\boxed{N_S(\lambda,z) = \left(\frac{P_L(\lambda)\Delta t}{hc/\lambda}\right)\left(\sigma_{eff}(\lambda,z)n_c(z)R_B(\lambda)\Delta z\right)\left(\frac{A}{4\pi z^2}\right)\left(T_a^2(\lambda,z)E^2(\lambda,z)\right)\left(\eta(\lambda)G(z)\right)+N_B\Delta t}$$

Differential Absorption/Scattering Form

- For the laser with wavelength λ_{on} on the molecular absorption line

$$N_S(\lambda_{on},z) = N_L(\lambda_{on})\left[\beta_{sca}(\lambda_{on},z)\Delta z\right]\left(\frac{A}{z^2}\right)\exp\left[-2\int_0^z \overline{\alpha}(\lambda_{on},z')dz'\right]$$
$$\times\exp\left[-2\int_0^z \sigma_{abs}(\lambda_{on},z')n_c(z')dz'\right]\left[\eta(\lambda_{on})G(z)\right] + N_B$$

- For the laser with wavelength λ_{off} off the molecular absorption line

$$N_S(\lambda_{off},z) = N_L(\lambda_{off})\left[\beta_{sca}(\lambda_{off},z)\Delta z\right]\left(\frac{A}{z^2}\right)\exp\left[-2\int_0^z \overline{\alpha}(\lambda_{off},z')dz'\right]$$
$$\times\exp\left[-2\int_0^z \sigma_{abs}(\lambda_{off},z')n_c(z')dz'\right]\left[\eta(\lambda_{off})G(z)\right] + N_B$$

Differential Absorption/Scattering Form

- The ratio of photon counts from these two channels is a function of the differential absorption and scattering:

$$\boxed{\begin{aligned}\frac{N_S(\lambda_{on},z)-N_B}{N_S(\lambda_{off},z)-N_B} &= \frac{N_L(\lambda_{on})\beta_{sca}(\lambda_{on},z)}{N_L(\lambda_{off})\beta_{sca}(\lambda_{off},z)}\frac{\eta(\lambda_{on})}{\eta(\lambda_{off})} \\ &\times \exp\left\{-2\int_0^z\left[\overline{\alpha}(\lambda_{on},z')-\overline{\alpha}(\lambda_{off},z')\right]dz'\right\} \\ &\times \exp\left\{-2\int_0^z\left[\sigma_{abs}(\lambda_{on},z')-\sigma_{abs}(\lambda_{off},z')\right]n_c(z')dz'\right\}\end{aligned}}$$

$$\boxed{\Delta\sigma = \sigma_{abs}(\lambda_{on}) - \sigma_{abs}(\lambda_{off})}$$

Solutions of LiDAR Equation

- For scattering and fluorescence lidar equations

$$\beta(\lambda,\lambda_L,z) = \frac{N_S(\lambda,z) - N_B\Delta t}{\left(\frac{P_L(\lambda_L)\Delta t}{hc/\lambda_L}\right)\Delta z\left(\frac{A}{z^2}\right)\left(T(\lambda_L,z)T(\lambda,z)\right)\left(\eta(\lambda,\lambda_L)G(z)\right)}$$

$$n_c(z) = \frac{N_S(\lambda,z) - N_B\Delta t}{\left(\frac{P_L(\lambda)\Delta t}{hc/\lambda}\right)\left(\sigma_{eff}(\lambda)R_B(\lambda)\Delta z\right)\left(\frac{A}{4\pi z^2}\right)\left(\eta(\lambda)T_a^{\,2}(\lambda)E^2(\lambda,z)G(z)\right)}$$

Solutions of LiDAR Equation

- For resonance fluorescence lidar equation

$$N_S(\lambda,z) = \left(\frac{P_L(\lambda)\Delta t}{hc/\lambda}\right)\left(\sigma_{eff}(\lambda)n_c(z)R_B(\lambda)\Delta z\right)\left(\frac{A}{4\pi z^2}\right)\left(T_a^{\,2}(\lambda)E^2(\lambda,z)\right)\left(\eta(\lambda)G(z)\right) + N_B\Delta t$$

$$N_R(\lambda,z_R) = \left(\frac{P_L(\lambda)\Delta t}{hc/\lambda}\right)\left(\sigma_R(\pi,\lambda)n_R(z_R)\Delta z\right)\left(\frac{A}{z_R^{\,2}}\right)T_a^{\,2}(\lambda,z_R)\left(\eta(\lambda)G(z_R)\right) + N_B\Delta t$$

$$\frac{n_c(z)}{n_R(z_R)} = \frac{N_S(\lambda,z) - N_B\Delta t}{N_R(\lambda,z_R) - N_B\Delta t}\cdot\frac{z^2}{z_R^{\,2}}\cdot\frac{4\pi\sigma_R(\pi,\lambda)}{\sigma_{eff}(\lambda)R_B(\lambda)}\cdot\frac{T_a^{\,2}(\lambda,z_R)G(z_R)}{T_a^{\,2}(\lambda,z)E^2(\lambda,z)G(z)}$$

$$\boxed{n_c(z) = n_R(z_R)\frac{N_S(\lambda,z) - N_B\Delta t}{N_R(\lambda,z_R) - N_B\Delta t}\cdot\frac{z^2}{z_R^{\,2}}\cdot\frac{4\pi\sigma_R(\pi,\lambda)}{\sigma_{eff}(\lambda)R_B(\lambda)}\cdot\frac{1}{E^2(\lambda,z)}}$$

Solutions of LiDAR Equation

- For differential absorption lidar equation

$$n_c(z) = \frac{1}{2\Delta\sigma}\frac{d}{dz}\left\{ \begin{aligned} &\ln\left[\frac{N_L(\lambda_{on})\beta_{sca}(\lambda_{on},z)}{N_L(\lambda_{off})\beta_{sca}(\lambda_{off},z)}\frac{\eta(\lambda_{on})}{\eta(\lambda_{off})}\right] \\ &-\ln\left[\frac{N_S(\lambda_{on},z)-N_B}{N_S(\lambda_{off},z)-N_B}\right] \\ &-\left[\bar{\alpha}(\lambda_{on},z')-\bar{\alpha}(\lambda_{off},z')\right] \end{aligned} \right\}$$

$$\Delta\sigma = \sigma_{abs}(\lambda_{on}) - \sigma_{abs}(\lambda_{off})$$

Summary

- Lidar equation relates the received photons (power) to the transmitted photons (power), properties of scatters, medium transmissions, and system efficiencies. It is the basic equation governing the lidar field.

- Physical processes in lidar equation include

  elastic/inelastic scattering,

  absorption/differential absorption,

  fluorescence/resonance fluorescence,

  Doppler effect (shift and broadening),

  Boltzmann distribution, and extinction.

- Different forms of lidar equations and their solutions are presented for different physical interactions.